Underlying catastrophes: umbilics and pattern formation

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Abstract

Bifurcation theory provides a very general means to *classify* the local changes in numbers of zeros of vector fields, but not a general means to *find where* a given bifurcation occurs, at least at higher codimensions. Instead, it turns out, these bifurcations can be found by looking for their underlying catastrophes. Here I show that the concept of underlying catastrophes can be extended to the umbilics. The umbilics are important in opening up qualitatively different forms of bifurcations beyond the 'corank 1' catastrophes of folds, cusps, swallowtails, etc. An example is given showing how four zeros of a vector field bifurcating from a single point, may do so either via a 3-parameter swallowtail catastrophe involving equilibria of similar stabilities, or via a 4-parameter umbilic catastrophe involving equilibria of opposing stabilities. This opens an avenue to studying spatiotemporal pattern formation around high codimension bifurcation points, and I conclude with some illustrative examples.

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1 Introduction

The number of stationary states of a dynamical system can change with parameters at *singularities*, creating *bifurcations* in the local topology. How can we map out the sets on which these events occur? The classification of singularities and bifurcations generally relies on local expressions that require one first to know the location of a given singularity. Surprisingly, perhaps, this location problem is not solvable in general. Here I describe how the concept of an *underlying catastrophe* provides solvable conditions to find the location of a broad class of singularities and bifurcations in vector fields. I extend the method of underlying catastrophes developed in [19, 20] to corank 2 *umbilics*, and show how these ideas help the discovery of bifurcations in wider contexts, such as spatiotemporal problems.

For a system governed by a scalar or gradient function, the locations of singularities can be found by solving simply for points where certain derivatives vanish, e.g. using catastrophe theory [31, 27]. This is not complete, however, as the less well known fact is that even gradient vector field are not fully described by catastrophe theory, as a field $\mathbf{F} = \nabla\phi$ may exhibit unfoldings *not* captured by Thom's catastrophe unfoldings of the potentials ϕ [16]. The situation is even worse for general vector fields, namely that, although there are general classifications of their bifurcations, e.g. [2, 8, 10], explicit conditions to identify them only exist for some low codimensions (e.g. [2, 17, 21, 30, 9]), and even less is known for systems on spatiotemporal domains, such as reaction-diffusion or other partial differential equations.

The concept of underlying catastrophes reduces these generally intractable problems to something pragmatic for the purposes of calculation. In essence, the method identities points where multiple stationary points of a system collide, in a manner equivalent to the *elementary catastrophes* of scalar functions. I set out the relation to standard theory here, by using Boardman's already somewhat practical symbolic classification scheme, to set out rigorous relations to Thom's geometric theory. While overcoming a huge practical barrier — one that has become more obvious in an age dominated by computation — this also shows the still far-reaching power of the theory set out by Thom and his contemporaries (including Morin, Whitney, Mather, Zeeman, see e.g. [12, 23, 24, 26, 33, 36]). Hopefully, in doing so, I also remind a modern audience of this beautiful period in the genesis of modern bifurcation theory. I set this out as far as possible in a practical, minimally technical, manner, for the user of applied bifurcation theory, rather than the theorist. This also serves to demonstrate, however, how easy it is to identify underlying catastrophes by the procedure in section 2, versus singularities by the Thom-Boardman procedure in section 3.

I must first recall here the basic idea from [19, 20], which was defined only for corank 1 singularities. I will simplify some of the concepts and extend them beyond corank 1 to find underlying analogues of the *umbilic* catastrophes. I demonstrate this by contrasting swallowtail versus umbilic catastrophes of 4 stationary states in a planar vector field, and then giving examples of pattern formation around underlying catastrophes in reaction-diffusion problems up to codimension four.

At present the knowledge of how to extend underlying catastrophes to the umbilics is limited. Specifically, conditions are given here for the simplest class of underlying umbilic catastrophes, of codimension 4, in two or three dimensions. My hope is that these will point the way to a general theory that will give the final extension to all codimensions and all higher dimensions, in future work. The extension to umbilics is non-trivial, however, and already the Thom-Boardman classification cannot distinguish all types of umbilic [35]. These tentative steps are important, however, as the umbilics open up a fundamentally different class of singularities beyond the corank 1 cases, both theoretically, as they relate to an entirely different family of symmetry groups [3, 24], and practically, because they were instrumental in establishing the applications of catastrophes to physics, see e.g. [5, 14, 27, 35].

The difference between *singularities* and *underlying catastrophes* can perhaps be understood as one of perspective. While the former attempt to classify a *complete* family of possible equivalence classes of certain singular scenarios, the latter give up *completeness* in favour of those that can be found in practice. These elements are contained in the codimension of the singularity, and the number of parameters in its unfolding, versus the number of conditions needed to characterise it. Underlying catastrophes explicitly reduce the problem to the minimal number of conditions, matching the codimension, with the 'fullness' conditions for their solvability. This reduction is not trivial, and excludes some singularities contained in the Thom-Boardman classification. Even the Thom-Boardman classification is known to not entirely distinguish all singularities, particularly the umbilics [35]. The 'fullness' condition is stronger than genericity/non-degeneracy, reducing the system of equations necessary to find a given singularity to something solvable, for example it reduces 95 million Thom-Boardman calculations to identify a butterfly singularity in a 4 dimensional system, to just 8 solvable calculations that can be solved explicitly to *locate* that butterfly. One uses an *underlying catastrophe* to identify where an event occurs, and can then apply singularity/bifurcation theory to formerly classify that event.

I begin in section 2 by summarizing the conditions used to find underlying catastrophes, including a summary of the new results on umbilics. This is set out in more detail in section 3 by relating it to Thom-Boardman's singularity classification, with detail on how that method can be applied. The remaining sections are illustrations of the methodology, of using the '$\mathcal{B}$-$\mathcal{G}$' conditions to find and identify underlying catastrophes, and understand some of their characteristics. In section 4 I show two vector fields each with 4 steady states bifurcating from two different underlying catastrophes, with example calculations. In section 5 I simulate the deffering patterns found around two underlying catastrophes in a reaction-diffusion problem, as illustrations of their effect on pattern formation, mainly as inspiration for future works. Some closing remarks are made in section 6.

2 Finding underlying catastrophes: $\mathcal{B}$-$\mathcal{G}$ conditions

Take a smooth vector field $\mathbf{F} : \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n$ with components $\mathbf{F} = (f_1, f_2, ..., f_n)$, which are functions of a variable $\mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}^n$ and parameter $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_r) \in \mathbb{R}^r$. Denote the gradient operator $\nabla = \frac{\partial}{\partial \mathbf{x}}$, and an extended gradient operator $\Box = (\frac{\partial}{\partial \mathbf{x}}, \frac{\partial}{\partial \boldsymbol{\alpha}})$.

Our interest concerns steady states of a vector field, so let us assume we are concerned with a point p where $\mathbf{F} = 0$.

If the Jacobian determinant $\mathcal{B}_1 = |\nabla (f_1, , f_2, ..., f_n)|$ vanishes at the point of interest p, then that is a singularity of $\mathbf{F}$ (more precisely of the map $\mathbf{F} : \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n$). We then seek solvable quantities that characterise that singularity such that we can find its location in the space of $(\mathbf{x}, \boldsymbol{\alpha}) \in \mathbb{R}^n \times \mathbb{R}^r$.

2.1 Corank 1

The vanishing of $\mathcal{B}_1$ indicates that the vectors $\nabla f_1, ..., \nabla f_n$, are linearly dependent at the point of interest. If the Jacobian has corank 1, we can typically assume that any $n-1$ of the gradient vectors $\nabla f_1, ..., \nabla f_n$, are linearly independent. Under this assumption, the degeneracy of the point can then be characterised by a sequence of determinants

$$\mathcal{B}_i = |\nabla (\mathcal{B}_{i-1}, f_2, ..., f_n)| \quad \text{with} \quad \mathcal{B}_1 = |\nabla(f_1, f_2, ..., f_n)| \ . \tag{1}$$

If the conditions $\mathbf{F} = 0$ and $\mathcal{B}_1 = ... = \mathcal{B}_r = 0 \neq \mathcal{B}_{r+1}$ are solvable at some point $\mathbf{x}_*$ and parameter value $\boldsymbol{\alpha}_*$, then we say that point is an *underlying catastrophe* of codimension r. These correspond to a fold ($r = 1$), cusp

($r = 2$), swallowtail ($r = 3$), etc. as per Thom's elementary catastrophes. To check that these conditions are indeed uniquely solvable we need to check that a further set of determinants,

$$\mathcal{G}_{i,K(i)} = \left|\Box\left(f_1, ..., f_n, \mathcal{B}_1, ..., \mathcal{B}_{i,K(i)}\right)\right| \ , \tag{2}$$

are non-vanishing for all $k_j = 1, ..., n$, over $j = 1, ..., i-1$, where $K(i)$ denotes the index string

$$K(i) = k_1...k_{i-1} \ ,$$

and where the functions $\mathcal{B}_{i,K(i)} \equiv \mathcal{B}_{i,k_1...k_{i-1}}$ are determinants

$$\mathcal{B}_{i,K(i)} = \left|\nabla\left(f_1, ..., f_{h-1}, \mathcal{B}_{i-1,K(i-1)}, f_{h+1}, ..., f_n\right)\right| \ , \tag{3}$$

with $h = k_{i-1}$. Let us explain these briefly here, with a more complete explanation in section 3.

A singularity occurs where the Jacobian of $\mathbf{F}$ is singular, i.e. $\mathcal{B}_1 = 0$, as $\nabla\mathbf{F}$ spans a space of dimension $n-1$. The vanishing of each higher order determinant $\mathcal{B}_{i,k_1...k_{i-1}}$ detects a degeneracy of higher order, where the space spanned by $\mathbf{F}$ and the lower order $\mathcal{B}_{i-1,k_1...k_{i-2}}$ has dimension $n-1$. With care, we can use any of these conditions to locate a singularity, but typically, at a codimension r singularity, all conditions $\mathcal{B}_{i,k_1...k_{i-1}}$ vanish for all $i = 1, ..., r$. There are $\frac{1-n^r}{1-n} \sim n^r$ such conditions, far too many to be solvable in general, but many of these equations turn out to be either redundant or equivalent, such than the vanishing of any of the $\mathcal{B}_{i,k_1...k_{i-1}}$ imply the vanishing of all of them, as proven in [20].

The underlying catastrophes use this fact to reduce the localization problem to a solvable set of conditions. Provided the determinants (2) are non-vanishing, then, in the space of $(\mathbf{x}, \boldsymbol{\alpha})$, there is sufficient independence between the vectors $\mathbf{F}$ and $\Box\mathcal{B}_{i,k_1...k_{i-1}}$ that we do not need to consider all the possible combinations of k_js, instead we can make one particular choice for all the $k_1...k_{i-1}$ at each i. We identify these as the set of functions $\mathcal{B}_i := \mathcal{B}_{i,k_1...k_{i-1}}$ for a fixed number string $k_1...k_{i-1}$. In (1) we make the obvious choice $\mathcal{B}_i \equiv \mathcal{B}_{i,1...1}$ for convenience. For a given problem, any particular choice of the k_js may provide a more efficient set of conditions $\mathcal{B}_i = 0$ to solve for $i = 1, ..., r$. To find a codimension r singularity in n dimensions, with r having (at least) r parameters, we now have a solvable set of n conditions $\mathbf{F} = 0$ and r conditions $\mathcal{B}_1 = ... = \mathcal{B}_r = 0$. The proof that any choice of the functions $\mathcal{B}_i$ from among the $\mathcal{B}_{i,k_1...k_{i-1}}$ is in [20].

Thus an underlying catastrophe is a point or set of points where certain geometrical conditions are satisfied, it is not a topological class, because it makes no consideration of the local topology of the vector field. However, once the point is located, it can be properly classified using local singularity theory. The non-degeneracy conditions $\mathcal{G}_{i,k_1...k_{i-1}} \neq 0$ are rather numerous, but we need only evaluate them to check they are non-vanishing.

2.2 Corank 2

If the determinant of the Jacobian of $\mathbf{F}$ at p (where $\mathbf{F} = 0$) has corank 2, then we enter the important class of *umbilics*, which can only exist in two or more dimensions. They are far more complicated than the codimension 1 singularities. Here we make some first steps in extending the $\mathcal{B}$-$\mathcal{G}$ conditions to the umbilics.

If the corank of $\nabla\mathbf{F}$ is 2 then the singularity is known, from Thom's elementary catastrophe theory, to be of codimension 3 at least in a gradient vector field. If $\mathbf{F}$ is not a gradient field we can expect umbilics to be of codimension 4 at least ('releasing' the one constraint of being a gradient field). We will see here how this follows since $\nabla\mathbf{F}$ has corank 2 if and only if at least 4 of its corank 1 minors vanish.

Though the corank of $\nabla\mathbf{F}$ is 2, we can typically assume that any $n-2$ of the gradient vectors $\nabla f_1, ..., \nabla f_n$, are linearly independent. We will now need to deal with vectors of length $n-1$, so let us denote a vector $\mathbf{v}$ with the j^{th} component deleted by $\mathbf{v}_{-j} = (v_1, ..., v_{j-1}, v_{j+1}, ..., v_n)$. Then we can define determinants

$$\mathcal{B}^2_{1,k} = \left| \frac{\partial \mathbf{F}_{-u}}{\partial \mathbf{x}_{-v}} \right| , \qquad \begin{array}{l} k = v + (u-1)n, \\ u, v = 1, ..., n, \end{array} \tag{4}$$

which are just the corank 1 minors of the Jacobian of $\mathbf{F}$. An umbilic occurs at a point where $\mathbf{F} = 0$ and all of these corank 1 minors vanish. In a gradient vector field, three of these vanishing is sufficient to define the elementary hyperbolic or elliptic umbilic catastrophes, see e.g. [15, 27]. For a general (i.e. non-gradient) vector field, I will argue in section 3.2 that four of these vanishing is sufficient to define corresponding codimension 4 underlying umbilic catastrophes, say $\mathcal{B}^2_{1,1} = \mathcal{B}^2_{1,2} = \mathcal{B}^2_{1,3} = \mathcal{B}^2_{1,4} = 0$, provided that all

$$\mathcal{G}^2_{1,k_1k_2k_3k_4} = \left| \left(\Box \left(f_1, ..., f_n,\ \mathcal{B}^2_{1,k_1},\ \mathcal{B}^2_{1,k_2},\ \mathcal{B}^2_{1,k_3},\ \mathcal{B}^2_{1,k_4} \right) \right) \right| , \tag{5}$$

are non-vanishing, for all $k_1, k_2, k_3, k_4 \in 1, ..., n^2$ with $k_1 \neq k_2 \neq k_3 \neq k_4$.

I will also show that to extend this to the next higher codimension will involve minors that involve functions of the form

$$\mathcal{B}^2_{2,k_1k_2} = \left| \nabla \left(\mathcal{B}^2_{1,k_1},\ \mathcal{B}^2_{1,k_2},\ f_3, ..., f_n \right) \right| , \tag{6}$$

and ongoing extensions along these lines are discussed in section 3.2.

Before expanding on the arguments above and giving some examples, let me offer one practical note (for both corank 1 and corank 2), concerning the derivative operator $\square$ and the parameters α_i. At each stage we select a set of parameters $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_r)$ to work in, but a given model may have more than r parameters. As is usual in any bifurcation analysis, to choose an appropriate set of parameters $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_r)$ from all those available requires inspection. The conditions $\mathcal{B}_i = 0$ might be solvable only in certain subsets of r parameters, and we choose to write any viable set as the list of parameters $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_r)$. The method accounts for this of course, as the conditions $\mathcal{G}_{i,k} \neq 0$ indicate that the conditions $\mathcal{B}_i = 0$ can be solved in the given choice of parameters. The conditions $\mathcal{G}_{i,k} \neq 0$ single out codimension r events that are solvable (i.e. locatable) in r parameters. These conditions are not highly restrictive, however they do rule out classes with certain symmetries or conserved quantities, such as Hamiltonian systems. It is not yet know whether underlying catastrophes can be defined in such classes, but certain special cases are trivial. In [19] it was shown that $\mathcal{G}_{i,k} = 0$ can indicate merely that certain variables in the system are redundant, so removing a dimension gives a solvable problem. For gradient systems the underlying catastrophe *is* just an elementary catastrophe. The elaborations on the method given here may help in developing such extensions in future work.

As one important final note from [19, 20], the definition of the underlying catastrophes is independent of the choice of coordinates. Moreover the conditions $\mathcal{G}_{i,k} \neq 0$ ensure that the definition is independent of the choice of Jacobian $\mathcal{B}_{i,k} \to \mathcal{B}_i$ for each i.

3 The procedure in detail: ranks and classes

The geometrical insight behind the $\mathcal{B}$-$\mathcal{G}$ conditions was given in [19], and in [20] it was shown that, under the necessary solvability conditions $\mathcal{G}_{i,k} \neq 0$, the conditions $\mathcal{B}_i = 0$ constitute an efficient implementation of the classification developed by Thom, Boardman, and Morin [4, 23, 26, 31] (see also [22, 24]).

Here I summarise how the $\mathcal{B}$-$\mathcal{G}$ conditions for finding underlying catastrophes relate to the Thom-Boardman procedure for classifying singularities, and show how this allows us to extend the idea to corank 2, i.e. the umbilics, motivating the conditions given in section 2.2.

To simplify notation we can assume that any function is defined for all $(\mathbf{x}, \boldsymbol{\alpha})$, but when we discuss a specific value (or rank) of a function then we refer to a specific point $(\mathbf{x}_*, \boldsymbol{\alpha}_*)$, namely the singularity of interest.

We begin simply by identifying a point where the vector field $\mathbf{F}$ has a stationary state or zero, that is, a point $(\mathbf{x}_*, \boldsymbol{\alpha}_*)$ where

$$\mathbf{F} = 0 \ . \tag{7}$$

Let $\mathcal{B}_1$ denote the Jacobian determinant of $\mathbf{F}$,

$$\mathcal{B}_1 = |\nabla F| \ . \tag{8}$$

There is a singularity at $(\mathbf{x}_*, \boldsymbol{\alpha}_*)$ if $\mathcal{B}_1 = 0$ there.

To characterise this singularity we can define a sequence of symbols $\Delta^{i_1}\mathbf{F}$ through to $\Delta^{i_j}...\Delta^{i_1}\mathbf{F}$ that determine its rank and codimension. We begin by defining the symbol $\Delta^{i_1}\mathbf{F}$ in terms of a vector $\mathbf{B}_1^{i_1}$ as

$$\begin{aligned} \Delta^{i_1}\mathbf{F} &= \left(\mathbf{F},\ \mathbf{B}_1^{i_1}\right) \qquad \text{where} \\ \mathbf{B}_1^{i_1} &= (m_{1,1}^{i_1},\ m_{1,2}^{i_1},\ ...,\ m_{1,N_1}^{i_1})\ , \end{aligned} \tag{9}$$

with $m_{1,j}^{i_1}$, $j = 1, ..., N_1$, denoting the $(n - i_1 + 1) \times (n - i_1 + 1)$ minors of $\nabla\mathbf{F}$. Next we define $\Delta^{i_2}\Delta^{i_1}\mathbf{F} = \left(\Delta^{i_1}\mathbf{F}, m_{2,1}^{i_2}, m_{2,2}^{i_2}, ...\right)$, where $m_{2,j}^{i_2}$ are the $(n - i_2 + 1) \times (n - i_2 + 1)$ minors of $\nabla(\Delta^{i_1}\mathbf{F})$, and proceed iteratively, next defining $\Delta^{i_3}\Delta^{i_2}\Delta^{i_1}\mathbf{F} = \left(\Delta^{i_2}\Delta^{i_1}\mathbf{F}, m_{3,1}^{i_3}, m_{3,2}^{i_3}, ...\right)$, where $m_{3,j}^{i_3}$ are the $(n - i_3 + 1) \times (n - i_3 + 1)$ minors of $\nabla(\Delta^{i_2}\Delta^{i_1}\mathbf{F})$, and so on, giving generally

$$\begin{aligned} \Delta^{i_j}...\Delta^{i_1}\mathbf{F} &= \left(\Delta^{i_{j-1}}...\Delta^{i_1}\mathbf{F},\ \mathbf{B}_j^{i_j}\right) \qquad \text{where} \\ \mathbf{B}_j^{i_j} &= \left(m_{j,1}^{i_j},\ m_{j,2}^{i_j},\ ...,\ m_{j,N_j}^{i_j}\right)\ , \end{aligned} \tag{10}$$

with $m_{j,k}^{i_j}$ for $k = 1, ..., N_j$, being the $(n - i_j + 1) \times (n - i_j + 1)$ minors of $\nabla(\Delta^{i_{j-1}}...\Delta^{i_1}\mathbf{F})$.

Now we are going to work through these symbols, and check at each j^{th} step what the corank of $\nabla\Delta^{i_j}...\Delta^{i_1}\mathbf{F}$ is, and assign the next index i_j according to

$$i_{j+1} = \text{corank}\left(\nabla\Delta^{i_j}...\Delta^{i_1}\mathbf{F}\right)\ , \tag{11}$$

until we arrive at $i_{j+1} = 0$ for some j. Then we let $r = j$, and the symbol terminates as $\tau = i_1 i_2 ... i_r$ (and in the initial step '$j = 0$' we choose $i_1 =$ corank$(\nabla\mathbf{F})$). We call τ_j the j^{th} Boardman symbol, the symbols $\tau_1, ..., \tau_r$, form a non-increasing sequence $\tau_1 \geq \tau_2 \geq ... \geq \tau_r$, and we have the following.

Definition 3.0.1. *The Boardman symbol of* $\mathbf{F}$ *at* $\mathbf{x} = 0$ *is the sequence* $\tau = \tau_1, ..., \tau_r$*, such that each* $\nabla\Delta^{\tau_{j-1}}...\Delta^{\tau_1}\mathbf{F}(0)$ *has corank* τ_j *for* $j = 1, ..., r$*, (including that* $\nabla\mathbf{F}(0)$ *has corank* τ_1*), and the symbol* τ *is taken to terminate at* r *such that* $\tau_{r+1} = 0$*.*

Below we show how these calculations reduce to the $\mathcal{B}$-$\mathcal{G}$ conditions for the underlying catastrophes.

3.1 Corank 1

3.1.1 A zero of F

First take the trivial case of codimension $r = 0$, that is, a non-singular zero of $\mathbf{F}$, so the rank of $\nabla\mathbf{F}$ is n, i.e. $\mathcal{B}_1 = |\nabla\mathbf{F}| \neq 0$ and there is no singularity at the origin. Hence there is no Boardman symbol (or the symbol is just 0).

3.1.2 A fold

Now assume the first Boardman symbol is $\tau_1 = 1$. Then $\nabla\mathbf{F}$ has corank 1 and defines a singularity, with $\mathcal{B}_1 = |\nabla\mathbf{F}| = 0$.

If the second Boardman symbol is $\tau_2 = 0$, then the complete symbol is just $\tau = 1$, and defines a *fold*. Then the $(n + 1) \times n$ matrix $\nabla\Delta^1\mathbf{F} = (\nabla\mathbf{F}, \nabla\mathcal{B}_1)$ has rank n, so at least one of its $n \times n$ minors must be nonzero. We will look more closely at those minors in the next step.

We say that the fold is *full* if $(\nabla\mathbf{F}, \nabla\mathcal{B}_1)$ has rank n, and moreover $(\Box\mathbf{F}, \Box\mathcal{B}_1)$ has rank $n + 1$, so

$$\mathcal{G}_1 = |(\Box\mathbf{F}, \Box\mathcal{B}_1)| \ , \tag{12}$$

is non-vanishing. The property of being 'full' therefore implies that the conditions $\mathbf{F} = \mathcal{B}_1 = 0$ are solvable in $(\mathbf{x}, \alpha_1)$ by the implicit function theorem.

3.1.3 A cusp

If, instead, the second Boardman symbol is $\tau_2 = 1$, then $\nabla\mathbf{F}$ has corank 1, and moreover $\nabla\Delta^1\mathbf{F}$ has corank 1, so all of the $n \times n$ minors of $\nabla\Delta^1\mathbf{F}$ are zero. Those minors (with the exception of the minor $\nabla\mathbf{F}$ which we already know vanishes) are the functions $\mathcal{B}_{2,k}$ for $k = 1, ..., n$. Let

$$m^1_{2,k} = \mathcal{B}_{2,k} \quad \text{for} \quad k = 1, ..., n, \quad \text{and} \quad m^1_{2,n+1} = \mathcal{B}_1 \ . \tag{13}$$

Hence if $\tau_1 = \tau_2 = 1$ then $\mathcal{B}_1 = \mathcal{B}_{2,k} = 0$ for all $k = 1, ..., n$. The converse also holds, that if $\mathcal{B}_1 = \mathcal{B}_2 = 0$, then we have $\mathcal{B}_{2,k} = 0$ for all $k = 1, ..., n$, implying $\tau_1 = \tau_2 = 1$, as shown in [20].

Now if $\tau_3 = 0$ then we are done, and the singularity is a *cusp*, then the $2(n+1) \times n$ matrix $\nabla\Delta^1\Delta^1\mathbf{F} = \{\nabla\mathbf{F}, \nabla\mathcal{B}_1, \nabla\mathcal{B}_{2,1}, ..., \nabla\mathcal{B}_{2,n}, \nabla\mathcal{B}_1\}$ must have rank n, implying that at least one of the $n \times n$ minors of $\nabla\left(\Delta^1\Delta^1\mathbf{F}\right)$ is nonzero; again we will look more closely at these in the next step. We say the cusp is *full* if $(\Box\mathbf{F}, \Box\mathcal{B}_1, \Box\mathcal{B}_{2,k})$ has rank $n+2$, so

$$\mathcal{G}_{2,k} = |(\Box\mathbf{F}, \Box\mathcal{B}_1, \Box\mathcal{B}_{2,k})| \ , \tag{14}$$

is non-vanishing for all $k = 1, ..., n$.

3.1.4 A swallowtail

If instead $\tau_3 = 1$, then $\nabla\mathbf{F}$, $\nabla\Delta^1\mathbf{F}$, and $\nabla\Delta^1\Delta^1\mathbf{F}$ all have corank 1, so all of the $n \times n$ minors of $\nabla\Delta^1\Delta^1\mathbf{F}$ are zero. Recalling

$$\nabla(\Delta^1\Delta^1\mathbf{F}) = (\nabla\mathbf{F}, \nabla\mathcal{B}_1, \nabla\mathcal{B}_{2,1}, ..., \nabla\mathcal{B}_{2,n}, \nabla\mathcal{B}_1) \ ,$$

there are, therefore, $\frac{(2(n+1))!}{n!(n+2)!}$ of these minors, i.e. the binomial coefficient for choosing n rows from $2(n+1)$. One of these is $\mathcal{B}_1 = |\nabla\mathbf{F}|$, another $2n$ of them are the functions $\mathcal{B}_{2,1}, ..., \mathcal{B}_{2,n+1}$ repeated twice (because $\mathcal{B}_1$ is repeated twice in $\nabla\Delta^1\Delta^1\mathbf{F}$). Another n^2 are the determinants of Jacobian matrices in which some k_2^{th} row of $\nabla\mathbf{F}$ is swapped for one row $\nabla m^1_{2,k_1}$, with $k_1, k_2 = 1, ..., n$, and these are precisely the functions $\mathcal{B}_{3,k_1k_2}$, for $k_1, k_2 = 1, ..., n$. The remaining minors, say $m^1_{3,j}$ for $j = n^2+n+3, ..., \frac{(2(n+1))!}{n!(n+2)!}$, are the determinants of Jacobian matrices formed from $0 \leq d \leq n-2$ rows of $\nabla\mathbf{F}$ and $2 \leq d' \leq n$ rows from $(\nabla\mathcal{B}_1, \nabla\mathcal{B}_{2,1}, ..., \nabla\mathcal{B}_{2,n}, \nabla\mathcal{B}_1)$, but this has rank $n-1$, so all its $n \times n$ minors vanish. So we can enumerate all these minors as

$$m^1_{3,l(k_1,k_2)} = \mathcal{B}_{3,k_1k_2} \quad \text{for} \quad k_1, k_2 = 1, ..., n,$$
$$\text{with} \quad l(k_1,k_2) = k_1 + n(k_2-1) \ , \tag{15a}$$
$$m^1_{3,j} = \mathcal{B}_{2,j} \quad \text{for} \quad j = n^2+1, ..., n^2+n \ , \tag{15b}$$
$$m^1_{3,j} = \mathcal{B}_{2,j} \quad \text{for} \quad j = n^2+n+1, ..., n^2+2n \ , \tag{15c}$$
$$m^1_{3,j} = \mathcal{B}_1 \quad \text{for} \quad j = n^2+2n+1, ..., n^2+2n+2 \ , \tag{15d}$$
$$m^1_{3,j} = 0 \quad \text{for} \quad j = n^2+n+3, ..., \tfrac{(2(n+1))!}{n!(n+2)!} \ . \tag{15e}$$

So $\tau_1 = \tau_2 = \tau_3 = 1$ implies these minors all vanish, and hence $\mathcal{B}_1 = \mathcal{B}_{2,k_1} = \mathcal{B}_{3,k_1k_2} = 0$ for all $k_1, k_2 = 1, ..., n$. Again the converse also holds, that if $\mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B}_3 = 0$, then $\mathcal{B}_{2,k_1} = \mathcal{B}_{3,k_1k_2} = 0$ for all $k_1, k_2 = 1, ..., n$, implying $\tau_1 = \tau_2 = \tau_3 = 1$, as shown in [20].

Now if $\tau_4 = 0$ we are done, the singularity is a *swallowtail*, and the $(2(n+1)+\chi) \times n$ matrix

$$\nabla\Delta^1\Delta^1\Delta^1\mathbf{F} = \left\{\nabla\mathbf{F}, \nabla\mathcal{B}_1, \nabla m^1_{2,1}, ..., m^1_{2,n+1}, \nabla m^1_{3,1}, ..., \nabla m^1_{3,\chi}\right\}$$

where $\chi = \frac{(2(n+1))!}{n!(n+2)!}$, must have rank n since $\tau_3 = 0$, so at least one of its $n \times n$ minors must be nonzero; as usual, inspection of these is left to the next step. We say the swallowtail is *full* if $(\Box\mathbf{F}, \Box\mathcal{B}_1, \Box\mathcal{B}_{2,k_1}, \Box\mathcal{B}_{3,k_1k_2})$ has rank $n+3$, so

$$\mathcal{G}_{3,k_1k_2} = |(\Box\mathbf{F}, \Box\mathcal{B}_1, \Box\mathcal{B}_{2,k_1}, \Box\mathcal{B}_{3,k_1k_2})| \ , \tag{16}$$

is non-vanishing for all $k_1, k_2 = 1, ..., n$.

3.1.5 A butterfly, a wigwam, a star, . . .

And so on. At the next order, if $\tau_1 = \tau_2 = \tau_3 = \tau_4 = 1$, the minors include the functions $\mathcal{B}_{4,k_1k_2k_3}$, and we can define

$$m^1_{4,l(k_1,k_2,k_3)} = \mathcal{B}_{4,k_1k_2k_3} \quad \text{for} \quad k_1, k_2, k_3 = 1, ..., n, \qquad \text{where} \quad l(k_1,k_2,k_3) = k_1 + n(k_2-1) + n^2(k_3-1) \ , \tag{17}$$

while the remaining $m^1_{4,j}$ for $j = n^3+1, ..., \frac{(2(n+1)+\chi)!}{n!((2(n+1)+\chi)-n)!}$, consist of the functions $\mathcal{B}_1$, $\mathcal{B}_{2,k_1}$, $\mathcal{B}_{3,k_1k_2}$, for $k_1, k_2 = 1, ..., n$, as well as determinants of Jacobian matrices formed from $0 \leq d \leq n-2$ rows of $\nabla\mathbf{F}$ and $2 \leq d' \leq n$ rows from $\left(\nabla\mathcal{B}_1, \nabla m^1_{2,1}, ..., \nabla m^1_{2,n+1}, \nabla m^1_{3,1}, ..., \nabla m^1_{3,\chi}\right)$, whose minors must vanish. If $\tau_4 = 0$ then the singularity is a *butterfly*, and it is full if $(\Box\mathbf{F}, \Box\mathcal{B}_1, \Box\mathcal{B}_{2,k_1}, \Box\mathcal{B}_{3,k_1k_2}, \Box\mathcal{B}_{4,k_1k_2k_3})$ has rank $n+4$, hence if

$$\mathcal{G}_{4,k_1k_2k_3} = |(\Box\mathbf{F}, \Box\mathcal{B}_1, \Box\mathcal{B}_{2,k_1}, \Box\mathcal{B}_{3,k_1k_2}, \Box\mathcal{B}_{4,k_1k_2k_3})| \tag{18}$$

is non-vanishing for all $k_1, k_2, k_3 = 1, ..., n$.

At each successive symbol of length r, the vanishing of minors is equivalent to the vanishing of all of the quantities $\mathbf{F}$, $\mathcal{B}_1$, $\mathcal{B}_{2,k_1}$, ... $\mathcal{B}_{r,k_1...k_{r-1}}$. Each case is full if $\left(\Box\mathbf{F}, \Box\mathcal{B}_1, \Box\mathcal{B}_{2,k_1}, ..., \Box\mathcal{B}_{r,k_1...k_{r-1}}\right)$ has rank $n+r$, so

$$\mathcal{G}_{4,k_1k_2k_3} = \left|\left(\Box\mathbf{F}, \Box\mathcal{B}_1, \Box\mathcal{B}_{2,k_1}, ..., \Box\mathcal{B}_{r,k_1...k_{r-1}}\right)\right| \ , \tag{19}$$

is non-vanishing for all $k_1, ..., k_{r-1} = 1, ..., n$, and then the numerous minors, which have already been reduced to the set of functions $\mathcal{B}_1$, $\mathcal{B}_{2,k_1}$, ... $\mathcal{B}_{r,k_1...k_{r-1}}$, reduce further to the minimal set of determinants $\mathcal{B}_1$, $\mathcal{B}_2$, ... $\mathcal{B}_r$.

This procedure can be carried out to arbitrarily high codimension r, providing a solvable algebraic system of n equations $\mathbf{F} = 0$, and r equations $\mathcal{B}_1 = ... = \mathcal{B}_r = 0$, to locate a given underlying catastrophe (and hence the singularity) in $(\mathbf{x}, \boldsymbol{\alpha}) \in \mathbb{R}^n \times \mathbb{R}^r$.

3.2 Corank 2

Say $\mathbf{F} = 0$ and $\mathcal{B}_1 = 0$, but $\tau_1 = \text{corank}(\nabla\mathbf{F}) = 2$. We will extend the arguments above as simply as possible, namely seeking to identify the minors at each codimension r, and reduce them to a set of functions that is solvable in r parameters.

The symbol $\Delta^2\mathbf{F}$ consists of $\mathbf{F}$ itself, along with the $(n-1) \times (n-1)$ minors of $\nabla\mathbf{F}$. So define

$$\begin{aligned} \Delta^2\mathbf{F} &= \left(\mathbf{F},\ \mathbf{B}_1^2\right) \qquad \text{where} \\ \mathbf{B}_1^2 &= (m_{1,1}^2,\ m_{1,2}^2,\ ...,\ m_{1,N_1}^2) \end{aligned} \tag{20}$$

where $m_{1,k}^2$ are the $(n-1) \times (n-1)$ minors of $\nabla\mathbf{F}$. One way to write these is

$$m_{1,k}^2 = \left| \frac{\partial(f_1, ..., f_{u-1}, f_{u+1}, ..., f_n)}{\partial(x_1, ..., x_{v-1}, x_{v+1}, ..., x_n)} \right| , \qquad \begin{array}{l} k = v + (u-1)n, \\ u, v = 1, ..., n, \end{array}$$

which we will use below, or more simply $m_{1,k}^2 = \left| \frac{\partial \mathbf{F}_{-u}}{\partial \mathbf{x}_{-v}} \right|$, again using the notation that $\mathbf{v}_{-j}$ denotes a vector $\mathbf{v}$ with its j^{th} component deleted,

$$\mathbf{v}_{-j} = (v_1, ..., v_{j-1}, v_{j+1}, ..., v_n) \quad \text{for any} \ \ j = 1, ..., n. \tag{21}$$

So for the simplest umbilic, with Boardman symbol $\tau = 2$ (i.e. $\tau_1 = 2$, $\tau_2 = 0$), we require $\Delta^2\mathbf{F} = 0$, which consists of $n + n^2$ conditions, and for non-degeneracy $\nabla\Delta^2\mathbf{F}$ should have full rank, that is

$$\text{rank}\left(\nabla\Delta^2\mathbf{F}\right) = n \ , \tag{22}$$

i.e. n of these $n + n^2$ should be linearly independent. We know only $n-2$ of $(\nabla f_1, ..., \nabla f_n)$ are linearly independent, so we require at least 2 of the $\nabla m_{1,k}^2$ to be linearly independent (to each other and to each ∇f_i). So if we look at the $n \times n$ minors,

$$\left| \left(n \text{ rows from } \nabla f_1, ..., \nabla f_n, \nabla \left| \frac{\partial \mathbf{F}_{-1}}{\partial \mathbf{x}_{-1}} \right|, ..., \nabla \left| \frac{\partial \mathbf{F}_{-u}}{\partial \mathbf{x}_{-v}} \right|, ... \right) \right| \ ,$$

we must find at least n that are non-vanishing.

Like the corank 1 calculations this involves more calculations than are necessary. If we define

$$\mathcal{B}^2_{1,k} = \left|\frac{\partial \mathbf{F}_{-u}}{\partial \mathbf{x}_{-v}}\right| \;, \qquad \begin{array}{l} k = v + (u-1)n, \\ u, v = 1, ..., n, \end{array} \tag{23}$$

the question is: how many of these must vanish to guarantee that they all vanish? The Jacobian $\frac{\partial \mathbf{F}}{\partial \mathbf{x}}$ has corank when the n vectors $\nabla f_1, ..., \nabla f_n$, span just $(n-2)$ dimensions. Typically this requires 4 conditions, which can be seen as follows. Some $n-2$ of these vectors span a space $\mathcal{T}$, say $\nabla f_1, ..., \nabla f_{n-2} \in \mathcal{T} \subset \mathbb{R}^{n-2}$, then it takes two 'rotations' to align each remaining ∇f_i with $\mathcal{T}$, hence four rotational constraints to give $\nabla f_{n-1}, \nabla f_n \in \mathcal{T}$. That any $n-1$ of these then spans $\mathcal{R}^{n-2}$ is then equivalent to the vanishing of the $(n-1)$-vector product $\nabla f_{i_1} \wedge \nabla f_{i_2} \wedge ... \wedge \nabla f_{i_{n-1}} = 0$ for all $i_1, ...i_{n-1} = 1, ..., n$ (in $n = 3$ dimensions this just means that $\nabla f_1, ..., \nabla f_{n-2} \in \mathcal{T} \subset \mathbb{R}$ and then all $\nabla f_{i_1} \wedge \nabla f_{i_2} = 0$ for all $i_1, i_2 = 1, 2, 3$, in $n = 2$ dimensions is just means that $\nabla f_{i_1} = 0$ for all $i_1 = 1, 2$). The components of these vector products are precisely the corank 1 minors (23).

Therefore we need any four of these minors to vanish, say $\mathcal{B}^2_{1,1} = \mathcal{B}^2_{1,2} = \mathcal{B}^2_{1,3} = \mathcal{B}^2_{1,4} = 0$. For this system to be solvable, which we define as the umbilic begin *full*, requires that

$$\begin{aligned} \mathcal{G}^2_{1,k_1k_2k_3k_4} &= \left|\left(\Box f_1, ..., \Box f_n, \text{ \& 4 rows from } \Box\left|\tfrac{\partial \mathbf{F}_{-1}}{\partial \mathbf{x}_{-1}}\right|, ..., \Box\left|\tfrac{\partial \mathbf{F}_{-u}}{\partial \mathbf{x}_{-v}}\right|, ...\right)\right| \\ &:= \left|\left(\Box\left(f_1, ..., f_n,\ \mathcal{B}^2_{1,k_1},\ \mathcal{B}^2_{1,k_2},\ \mathcal{B}^2_{1,k_3},\ \mathcal{B}^2_{1,k_4}\right)\right)\right| \;, \end{aligned} \tag{24}$$

is non-vanishing for all $k_1, k_2, k_3, k_4 \in 1, ..., n^2$ with $k_1 \neq k_2 \neq k_3 \neq k_4$. These conditions are demonstrated in examples of umbilic catastrophes in section 4.

This is just the first step in understanding the catastrophes that underly the umbilics of vector fields. A more complete extension requires further work, for a number of reasons. Firstly, it is known that the Boardman symbols are insufficient to fully classify the umbilics, for example they do not distinguish between umbilics of hyperbolic and elliptic type. In certain circumstances it seems that the non-degeneracy determinants in (24) do provide this distinction, but it has not yet been proven if this is always so. A full theory will require a deeper understanding of how the minors (23), their higher corank minors, and the $\mathcal{G}$ determinants, relate to the singularity classes of the singularities in Arnold's A, D, E, classification and the differential geometry of the umbilics, see e.g. [2, 35].

A first step in this will be to extend a crucial result proven in [20] for corank 1 singularities. When $\nabla\mathbf{F}$ has corank 1, it is possible to prove that the Boardman symbol $\tau = 1...1$ of length r, is equivalent to the vanishing of the r different determinants $\mathcal{B}_i$, $i = 1, ..., r$, if the underlying catastrophe is full, as shown in [20]. An important step in this is that the vanishing of one rank n minor $\mathcal{B}_i$ implies the vanishing of all rank n minors $\mathcal{B}_{i,k_1...k_{j-1}}$, for $i = 1, ..., r$, and all $k_i = 1, ..., n$.

It appears that further extending underlying catastrophes to the umbilics will require a similar result. It is clear enough that the vanishing of four of the $\mathcal{B}^2_{1,k}$ above implies the vanishing of all of them, and hence a corank 2 singularity. In the next step, if $\nabla\Delta^2\mathbf{F}$ does not have full rank then the next Boardman symbol is either $\tau_2 = 1$ or $\tau_2 = 2$. Taking the case $\tau_2 = 1$, define

$$\begin{aligned} \Delta^1\Delta^2\mathbf{F} &= \left(\mathbf{F},\ \mathbf{B}^2_1,\ \mathbf{B}^1_2\right) \qquad \text{where} \\ \mathbf{B}^1_2 &= (m^1_{2,1},\ m^1_{2,2},\ ...,\ m^1_{2,N_2}) \end{aligned} \tag{25}$$

with $m^1_{2,k}$ being the $n\times n$ minors of $\nabla\Delta^2\mathbf{F} = \nabla\left(\mathbf{F},\ \mathbf{B}^2_1\right)$. Since $n-2$ of the f_i are linearly independent, it would be natural to consider minors comprising $\mathbf{B}^1_2$ of the form of

$$\mathcal{B}^2_{2,k_1k_2} = \left|\nabla\left(\mathcal{B}^2_{1,k_1},\ \mathcal{B}^2_{1,k_2},\ f_3, ..., f_n\right)\right| \ . \tag{26}$$

However, it is not yet clear yet whether one or more of these vanishing is enough for all of them to vanish, i.e. whether one choice of k_1, k_2, is sufficient to give $\Delta^1\Delta^2\mathbf{F} = 0$, subject to non-degeneracy conditions in the form of determinants of the form

$$\mathcal{G}^2_{2,k_1k_2k_3k_4} = \left|\Box\left(\mathbf{F},\ \mathcal{B}^2_{1,k_1},\ \mathcal{B}^2_{1,k_2},\ \mathcal{B}^2_{1,k_3},\ \mathcal{B}^2_{1,k_4},\ \mathcal{B}^1_{2,k_1k_2}\right)\right| \ , \tag{27}$$

for $k_1, k_2, k_3, k_3 = 1, ..., n$, with $k_1 \neq k_2 \neq k_3 \neq k_4$. Solving this problem may provide the necessary theoretical step to define underlying catastrophes for all coranks, and is left to future work.

4 Four stationary states: swallows or umbilics?

One way the umbilics reveal themselves is in permitting bifurcations of stationary states of a fundamentally different character to corank 1 catastrophes. Le me illustrate this by considering two systems, both of which have up to 4 stationary states that can annihilate in a bifurcation, one by a swallowtail and one by an umbilic, the former of codimension 3, the latter codimension 4.

I will illustrate these underlying catastrophes in two vector fields $\mathbf{F} = (f, g)$ by considering the dynamical system $(\dot{x}, \dot{y}) = (f, g)$. Phase portraits summarising these are shown in fig. 1. The swallowtail is shown in fig. 1(left)(i), the umbilic in fig. 1(right)(i), at parameter and coordinate values to be found in equations (34) and (42), respectively. We shall find that both catastrophes involve two nodes and two saddles, but near the swallowtail the nodes have opposing stability, while near the umbilic they have equivalent stability. To illustrate this, the figure shows examples of nearby parameter values at which the system has (ii) 0, (iii) 2, or (iv) 4, stationary states, showing these differences in stability.

Example 4.1 (*A cubic-quadratic swallowtail)***.** Consider the planar vector field

$$(f, g) = (b + ax + y + x^3,\ c + x + y^2)\ , \tag{28}$$

in variables $\mathbf{x} = (x, y)$ and parameters $\boldsymbol{\alpha} = (a, b, c)$. Quite obviously, given the order of the polynomials, this can have up to 6 stationary states, so we cannot write their positions explicitly, but we can find expressions for the regions of parameter space that different families of them occupy, and find the underlying catastrophes that separate different families.

Conjecturing that a swallowtail occurs, let us evaluate the first three $\mathcal{B}_i$ determinants,

$$\begin{aligned} \mathcal{B}_1 &= |\nabla(f, g)| = 6x^2y + 2ay - 1\ , \\ \mathcal{B}_2 &= |\nabla(\mathcal{B}_1, g)| = 24xy^2 - 6x^2 - 2a\ , \\ \mathcal{B}_3 &= |\nabla(\mathcal{B}_2, g)| = 48y^3 - 72xy\ . \end{aligned} \tag{29}$$

There is a trick to solving catastrophe conditions like these, and in essence it involves inverting the roles of variables and parameters, to find the sets of catastrophes in the space of $\boldsymbol{\alpha}$, parameterised by $\mathbf{x}$. It becomes surprisingly easy to solve the conditions $0 = f = g = \mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B}_3$.

First, rather than trying to solve for the stationary states in (x, y) space, we can see simply that in (a, b, c) space the condition $(f, g) = 0$ implies

$$b = -ax - y - x^3\ , \qquad c = -x - y^2\ , \tag{30}$$

which is a volume in (a, b, c) space parameterised by (a, x, y).

These collide at folds, at which

$$\begin{aligned} & 0 = \mathcal{B}_1 = |\nabla(f, g)| = 6x^2y + 2ay - 1 \\ \Rightarrow\quad & a = (1 - 6x^2y)/2y\ , \end{aligned} \tag{31}$$

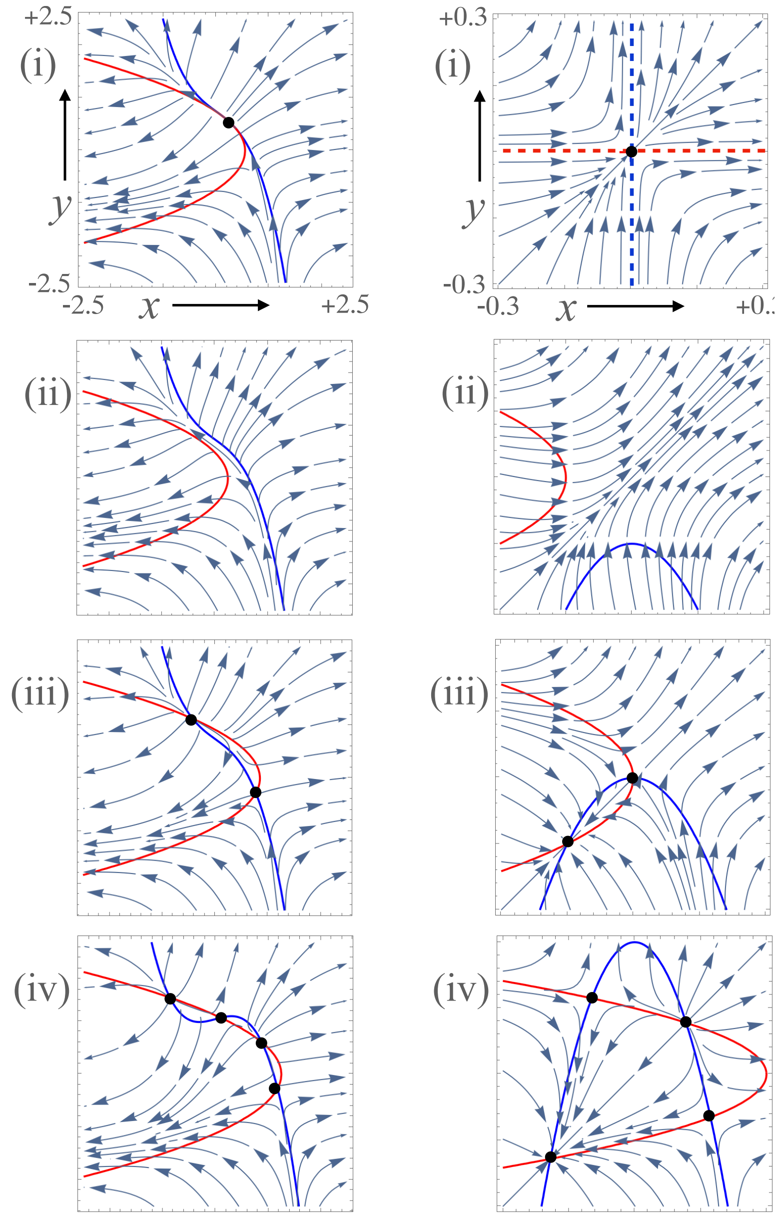

Figure 1: Phase portraits of Examples 4.1-4.2, showing (i) the catastrophe, breaking into: (ii) 0, (iii) 2, (iv) 4 stationary states. Nullclines $f = 0$ (blue curve) and $g = 0$ (red curve) are shown. Left, Ex.4.1, at parameter values (a, b, c) of: (i) $(\frac{5}{2^{6/5}3}, -\frac{35}{2^{4/5}27}, -\frac{5}{2^{8/5}3})$, (ii) $(0.7, -0.7, -0.2)$, (iii) $(0.7, -0.7, -0.8)$, (iv) $(-0.3, -1, -1.3)$. Right, Ex.4.2, at parameter values (a, b, c, d) of: (i) $(0, 0, 0, 0)$, (ii) $(1, 1, 1, 1)$, (iii) $(1, 0, 0, 1)$, (iv) $(\frac{1}{2}, -1, -1, \frac{1}{2})$.

hence defining with (30) a surface in (a,b,c) space, parameterized by (x,y).

The surfaces of folds have different branches, and they meet at creases of the surface, namely cusps

$$\begin{aligned} 0 = \mathcal{B}_2 = |\nabla(\mathcal{B}_1, g)| &= 24xy^2 - 6x^2 - 2a \\ &= 24xy^2 - 6x^2 - (1 - 6x^2 y)/y \\ \Rightarrow \qquad x &= 1/24y^4 \;, \end{aligned} \tag{32}$$

defining, with (30)-(31), curves in (a,b,c) space parameterized by y.

These curves themselves have branches that meet at the swallowtails, given by

$$\begin{aligned} 0 = \mathcal{B}_3 = |\nabla(\mathcal{B}_2, g)| &= 48y^3 - 72xy \\ \Rightarrow \qquad y &= 2^{-4/5} \;, \end{aligned} \tag{33}$$

giving finally, with (30)-(32), a point in (a,b,c) space, and hence the location of the swallowtail point at

$$(x,y) = (\tfrac{1}{2^{3/5}3}, \tfrac{1}{2^{4/5}}) \;, \quad (a,b,c) = (\tfrac{5}{2^{6/5}3}, -\tfrac{35}{2^{4/5}27}, -\tfrac{5}{2^{8/5}3}) \;. \tag{34}$$

In more general situations where this cannot be solved algebraically, these conditions can be solved numerically, using continuation methods to plot out the catastrophe sets in sections of $(\mathbf{x}, \boldsymbol{\alpha})$ space.

We have neglected to solve the accompanying conditions $\mathcal{G}_1 \neq 0$, $\mathcal{G}_{2,k_1} \neq 0$, $\mathcal{G}_{3,k_1 k_2} \neq 0$, in each step, but they are essential in ensuring that the solutions are valid (otherwise a different combination of the determinants $\mathcal{B}_{i,k_1 \dots k_{i-1}}$ may give a different solution to this choice of $\mathcal{B}_i$). I omit them for brevity, they are indeed non-vanishing here, and I shall give them just for the highest order. For the swallowtail there are 4 different $\mathcal{G}_{3,k_1 k_2}$ determinants to calculate, for $k_i = 1, 2$, and evaluated at the swallowtail point these give

$$\begin{aligned} \mathcal{G}_{3,k_1 k_2} &= |\square\,(f, g, \mathcal{B}_1, \mathcal{B}_{2,k_1}, \mathcal{B}_{3,k_1 k_2})| \\ &= 360 \begin{pmatrix} 2 & -2^{4/5} \\ 2^{3/5} & -2^{2/5} \end{pmatrix}_{k_1 k_2} \;. \end{aligned} \tag{35}$$

Since there is a swallowtail at the point given by (34), this implies that four stationary states should bifurcate from this point as the parameters (a,b,c) vary away from the values in (34). The left column of fig. 1 shows the flow of $(\dot{x}, \dot{y}) = (f, g)$ at the swallowtail, and three perturbations showing zero, two, or four stationary states. When there are two stationary states they consist of a saddle and a node, when there are four there are two saddles and two nodes, and both nodes have the same stability, in this case repelling. Obviously we can perform the calculations showing the stability of these stationary states explicitly using the analysis above, but I shall omit the standard calculations here for brevity.

A more standard approach to solving this problem is to reduce it to its proper one-dimensional normal form by transforming to coordinates parallel and orthogonal to the centre manifold of the bifurcation, but, of course, to do so, one must first know where the bifurcation occurs; that is the generally difficult problem that the underlying catastrophe allows us to solve quite easily. This being a planar problem of fairly low order, it is quite easy to solve $g = 0$ for x, substitute the resulting equation into $f = 0$ to obtain a 6^{th} order polynomial in y, and (provided certain transversality conditions with respect to the y coordinate), study the elementary catastrophes of this polynomial. These coincide with the catastrophes one obtains using the $\mathcal{B}$-$\mathcal{G}$ conditions, helping us understand what an underlying catastrophe is: it ignores the vectorial character of the problem to merely extract conditions revealing where stationary states of the system collide. For higher dimensional or higher order problems, obtaining a scalar polynomial for one of the variables may not be so easy, and typically is impossible, but I have kept to a relatively simple planar system here for ease of illustrating the resulting stationary states.

■

Example 4.2 (*A hyperbolic umbilic*)**.** Now consider the planar vector field

$$(f, g) = (b + ay + x^2,\ c + dx + y^2)\ , \tag{36}$$

in variables $\mathbf{x} = (x, y)$ and parameters $\boldsymbol{\alpha} = (a, b, c, d)$. Again, obviously from the order of the polynomials this can have up to 6 stationary states, and yet it has more parameters than (28). In this case we could find the stationary states explicitly, as the solutions of quartic equations, but of course these are unwieldy. Let us proceed similarly to the swallowtail and find how the situation differs.

First, to solve for the stationary states in (x, y) space, we can see that in (a, b, c, d) space the condition $(f, g) = 0$ implies

$$b = -ay - x^2\ , \qquad c = -dx - y^2\ , \tag{37}$$

which is a 4-volume in (a, b, c, d) space parameterized by (a, d, x, y).

To look for folds we then attempt to solve $0 = f = g = \mathcal{B}_1$, obtaining

$$0 = \mathcal{B}_1 = \left| \begin{pmatrix} 2x & a \\ d & 2y \end{pmatrix} \right| = 4xy - ad$$
$$\Rightarrow \qquad a = 4xy/d \; , \tag{38}$$

which with (37) defines a 3-volume in (a, b, c, d) space parameterized by (d, x, y).

These volumes will crease along cusps, where

$$0 = \mathcal{B}_2 = \left| \begin{pmatrix} 4y & 4x \\ d & 2y \end{pmatrix} \right| = 8y^2 - 4xd$$
$$\Rightarrow \qquad d = 2y^2/x \; , \quad a = 2x^2/y \; , \quad b = -3x^2 \; , \quad c = -3y^2 \; , \tag{39}$$

which with (37) and (38) define surfaces in (a, b, c, d) space parameterized by (x, y).

If those cusp surfaces have further degeneracies, could they be swallowtails? For this we need $\mathcal{B}_3 = 0 \neq \mathcal{G}_{3,k_1k_2}$, so first we solve

$$0 = \mathcal{B}_3 = \left| \begin{pmatrix} -4d & 16y \\ d & 2y \end{pmatrix} \right| = -24dy = -48y^3/x$$
$$\Rightarrow \qquad y = 0 \quad \Rightarrow \quad c = d = 0 \; . \tag{40}$$

Substituting $y = 0$ back in (39) gives a singular value for a, unless we also have $x = 0$, which would imply also $b = 0$: the fact that solving $\mathcal{B}_2 = 0$ results in not one constraint but multiple indicates that it is degenerate. This is confirmed by calculating $\mathcal{G}_{3,k_1k_2}$ with any three parameters from (a, b, c, d), for which we find $\mathcal{G}_{3,k_1k_2} = 0$. Hence this is not a swallowtail.

As usual for each catastrophe above we must evaluate the accompanying conditions $\mathcal{G}_1 \neq 0$, $\mathcal{G}_{2,k_1} \neq 0$, $\mathcal{G}_{3,k_1k_2} \neq 0$, again we omit them for brevity as they are straightforward calculations and are indeed nonzero.

The nature of the degeneracy that we have found with (40) is obvious if we calculate the Jacobian at this point, as all components vanish, so its corank is 2 (i.e. its rank is 0).

This implies that the underlying catastrophe may be an umbilic, the simplest of which is detected by the vanishing of $\mathcal{B}^2_{1,k}$ given by (26), and the non-vanishing of $\mathcal{G}^2_{1,1234}$ given by (27), so we calculate

$$\begin{aligned} 0 = \mathcal{B}^2_{1,k} &= (f_{,x}, f_{,y}, g_{,x}, g_{,y}) = (2x, z, d, 2y) \;, \\ \text{and} \qquad \mathcal{G}^2_{1,1234} &= \left|\square\left(f, g, \mathcal{B}^2_{1,1}, \mathcal{B}^2_{1,2}, \mathcal{B}^2_{1,3}, \mathcal{B}^2_{1,4}\right)\right| \\ &= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 0 & 0 \end{pmatrix} = 4 \;. \end{aligned} \tag{41}$$

Hence this implies the is an underlying umbilic catastrophe at

$$x = 0 \;, \quad y = 0 \;, \quad a = 0 \;, \quad b = 0 \;, \quad c = 0 \;, \tag{42}$$

from which emanate the cusps and folds found above.

This implies that four stationary states should bifurcate from the origin as we vary the parameters (a, b, c, d) away from their values in (42). The right column of fig. 1 shows the flow of $(\dot{x}, \dot{y}) = (f, g)$ at the umbilic, and three perturbations showing zero, two, or four stationary states. Similar to the swallowtail in the left column, when there are two stationary states they consist of a saddle and a node, when there are four there are two saddles and two nodes, but in this case the two nodes have opposing stability, one attracting and one repelling. Again we can perform the calculations showing the stability of these stationary states explicitly using the analysis above, but they are standard calculations and are omitted for brevity.

We can characterise this as a *hyperbolic umbilic*, most simply because if we let $d = a$ then we obtain a gradient system that corresponds to a hyperbolic umbilic in the classification of elementary catastrophes. Note that if we do fix $d = a$ then this system ceases to be *full* as an underlying catastrophe, as $\mathcal{G}^2_{1,1234}$ is then ill-defined, however as a gradient system this case then falls under the scope of elementary catastrophe theory.

■

The swallowtail and umbilic therefore exhibit three major differences, all highly relevant from the viewpoint of applications. The first is the number of parameters required to unfold them, though they involve the same number of stationary states. The second is that the curves of folds and cusps are therefore arranged around them somewhat differently, as revealed in the expressions of the fold and cusp sets above. The third difference is that, while in both the swallowtail and umbilic there are two saddles and two nodes (or foci for different parameters), the nodes are of the same stability around the swallowtail, and opposite stability around the umbilic.

The two examples certainly have other notable similarities and differences to distinguish them in applications, captured by the analysis above. Note, for instance, that both can exhibit coinciding pairs of folds, and we show an example for each system in fig. 2, that is, one set of parameters where two fold bifurcations occur at the same time, at different (x, y) values. What distinguishes the two systems in this regard is that the swallowtail can exhibits cusps at nearby parameters (where the two folds in fig. 2(left) approach and collide), while there are no cusps associated with the umbilic (the two folds in fig. 2(right) only collide at the umbilic point itself).

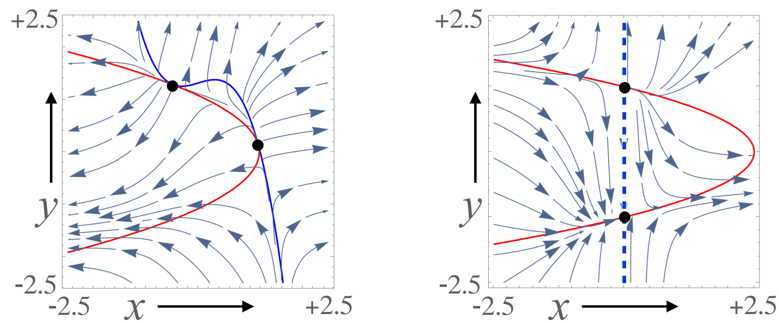

Figure 2: Phase portraits of Example 4.1 (left) and Example 4.2 (right), showing examples of of two co-existing folds. Parameter values are: left (a, b, c) equal to (i) $(-0.30, -1.32, -1.17)$ (to 2 decimal places.); right (a, b, c, d) equal to (i) $(0, 0, -1, 0.5)$.

5 Pattern forming around catastrophes

Singularity theory classifies the ways that systems can undergo local stability changes through bifurcations of stationary points, but lacks a general method to find the location of those singularities in multi-variable problems. Underlying catastrophes provide that practical method, and this opens up novel new areas to study the role of singularities.

One such application would be to pattern forming beyond spatial bifurcations. Turing [32] showed how the destabilisation of a stationary state by spatial diffusion terms led to pattern formation, but he also pre-saged bifurcation theory by proposing that, beyond mere pattern *formation*, nonlinearity would create the far more common phenomenon of *transition* from one pattern to another. Around higher order catastrophes we see that diffusive terms could destabilize a system in many different ways, enabling transitions as Turing envisaged. A detailed study is beyond the scope here, I will only illustrate the intriguing line of study with a few simulations, demonstrating the variety of patterns that may be found around such catastrophes. The model forms given may suggest to the reader other spatio-temporal applications where singularities might play a role, such as in organising phase boundaries in crystalline media, arranging wave patterns in neural models, or destabilising to induce sudden spread ecological systems. The examples here are chosen only to give relatively simple illustrations of a few catastrophes, and are only loosely based on the kind of models found in such systems, but motivation is drawn from models of nerve potentials [11], cell polarisation [25], chemical reaction models [28], and crystal phases [18] or convective instability [29].

In each example below we consider a simple reaction-diffusion equation of the form

$$\begin{aligned} \tfrac{\partial}{\partial t}u &= d_1\left(\tfrac{\partial^2}{\partial x^2}+\tfrac{\partial^2}{\partial y^2}\right)u+f(u,v)\;, \\ \tfrac{\partial}{\partial t}v &= d_2\left(\tfrac{\partial^2}{\partial x^2}+\tfrac{\partial^2}{\partial y^2}\right)v+g(u,v)\;, \end{aligned} \tag{43}$$

taking different 'reaction' functions (f,g) close to the site of underlying catastrophes, in such a way that the diffusive terms may induce instability.

Periodic boundary conditions are taken, on domain that will be specified below, along with initial conditions

$$u(x,y,0)=v(x,y,0)=1+0.2\sin(xy)\;, \tag{44}$$

solely for the purpose of illustration.

5.1 A cusp

Consider reaction functions

$$\begin{aligned} f(u,v) &= c_1+a_{11}u+a_{12}v-uv\;, \\ g(u,v) &= c_2+a_{21}u+a_{22}v-v^3\;, \end{aligned} \tag{45}$$

with $a_{11}=-4$, $a_{12}=-4.25$, $a_{21}=2$, $a_{22}=2$. Take periodic boundary conditions on a domain $(x,y)\in[-6\pi,+6\pi]^2$.

In the homogeneous system $(\dot u,\dot v)=(f,g)$, a cusp occurs where $f=g=B_1=B_2=0$ at $c_1=-0.358$, $c_2=0.183$. Varying parameters nearby we may have one stable stationary state, or a saddle-point surrounded by two stable stationary states, setting the scene for a diffusive-induced instability as diffusion is switched on in (43).

Figure 3 plots the system at c_i values where the system has a unique stable stationary state, but the addition of diffusion creates the patterns shown. In the left picture with $c_1 = c_2 = 0$, diffusion triggers a Turing instability (seen in the standard way by treating the spatial derivative as a $-k^2$ term for wave-like solutions, giving a pitchfork-like Turing bifurcation as d_1 and d_2 are increased).

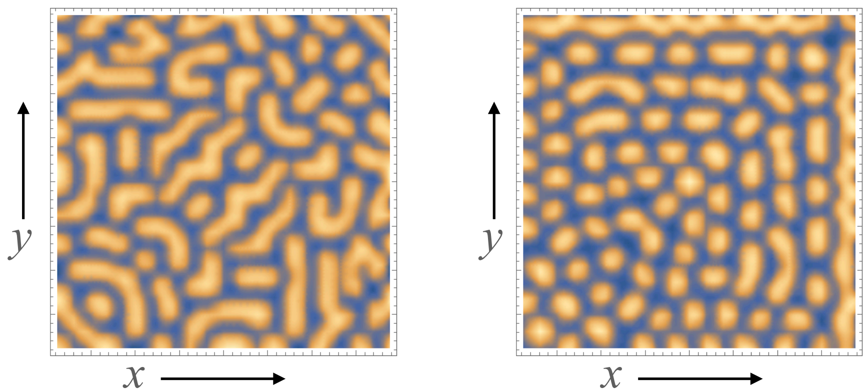

Figure 3: Patterns in the system (43) with (46), in a density plot of $u(x, y, t)$, with diffusion constants $d_1 = 3$, $d_2 = 0.1$. Left: $c_1 = c_2 = 0$, right: $c_1 = 0.2$, $c_2 = 0$. On domain $(x, y) \in [-6\pi, +6\pi]^2$. (Plots of $v(x, y, t)$ would show similar so I omit them)

In the right picture, with c_1 and c_2 nonzero, the system can undergo fold bifurcations, arranged around a cusp, and to study the diffusion instability requires more careful perturbative analysis, beyond our scope here but to be carried out in future work. Here we show that pattern formation similarly occurs, with a tendency towards spots rather than elongated 'worms' as c_1 (or c_2) increase.

5.2 A butterfly

Consider reaction functions

$$\begin{aligned} f(u, v) &= -u^3 - bu^2 - au - v - d \,, \\ g(u, v) &= -v^3 - v^2 - cv - u \,, \end{aligned} \tag{46}$$

for constants a, b, c, d, taking periodic boundary conditions on a domain $(x, y) \in [-18\pi, +18\pi]^2$.

The vector field (f, g) has an underlying butterfly catastrophe, where $f = g = \mathcal{B}_1 = \mathcal{B}_2 = \mathcal{B}_3 = \mathcal{B}_4 = 0$, at $(u, v) = (\frac{14}{27}, -\frac{2}{3})$, $(a, b, c, d) = (\frac{691}{243}, -\frac{23}{9}, 1, -\frac{5120}{19683})$. As we vary the parameters away from this, the butterfly bifurcates into up to 5 stationary states, and other nearby bifurcations lead to up to 7 stationary states. Calculations of the catastrophe conditions for similar functions can be found in [19, 20] (including necessary extra terms for the 7 stationary states to collapse into a star singularity). Some examples of this homogeneous system perturbed around the butterfly are illustrated in the homogenous system, that is $(\dot{u}, \dot{v}) = (f, g)$, in fig. 4.

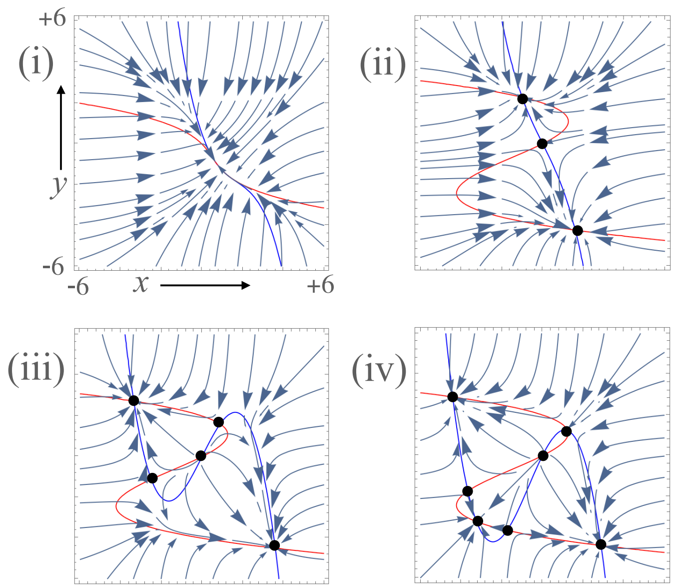

Figure 4: Flow of $(\dot{u}, \dot{v}) = (f, g)$ with (46), for: (i) $a = 2$, $b = 0$, $c = 1$; (ii) $a = 2.844$, $b = -2.555$, $c = 1$, $d = -0.260$; (iii) $a = -2$, $b = 0$, $c = -2$; (iv) $a = -2$, $b = 1$, $c = -2$.

Figure 5 shows steady patterns found in the diffusive system at different times.

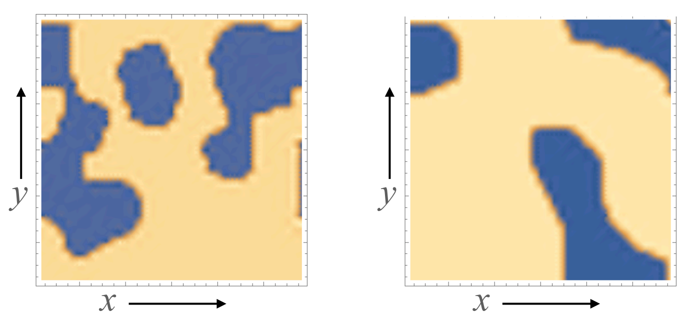

Figure 5: Patterns in the system (43) with (46), in a density plot of $u(x, y, t)$, for parameters (left) $d_1 = d_2 = 0.5$, $a = -2$, $b = 1$, $c = -2$, $d = 0$, $t = 100$, and (right) $d_1 = d_2 = 2$, $a = -2$, $b = 1$, $c = -2$, $d = 0$, $t = 100$. On domain $(x, y) \in [-18\pi, +18\pi]^2$.

6 Closing remarks

The study of the underlying catastrophes associated with different singularities and bifurcations is very new, being introduced in [19], motivated initially by tackling a reaction-diffusion 'wave-pinning' problem in [1], and with the first theory relating them to singularity theory via the Thom-Boardman classification in [20, 6]. Previous work has considered only underlying catastrophes for singularities where the vector field's Jacobian matrix has corank 1.

Extending this to the corank 2 case, the umbilics, is an important but much more challenging step, as the umbilics are themselves far richer, and also more problematic, for example the Thom-Boardman classification does not distinguish between them. Further work is required to understand the extension of underlying catastrophes to the umbilics more fully. Nevertheless, it is important to show that the umbilics can be identified, and moreover that they differ crucially from the corank 1 cases, in ways that are directly relevant to applications. I have shown here just one example, in which four steady states bifurcated from a single point, but in very different ways depending on whether the catastrophe was a swallowtail or an umbilic; they differ in the balance of stabilities, the geometry of local bifurcation curves, and even in the number of parameters unfolding the catastrophe.

There has been growing interest in recent years in studying the pattern-forming role of catastrophes and singularities in spatio-temporal problems, for instance their role in determining phase boundaries in crystalline media and chemical reactions [13, 7, 28, 18], crowd jamming [34], and wave forms in cell potentials [25, 1]. Already in his seminal paper [32], Turing discussed the role of nonlinear reaction functions in facilitating not just the onset of pattern-forming instability, but the transition — via bifurcations — between different patterns. I have briefly given examples here where the number of homogeneous steady states changes, altering the local stability, around high codimension catastrophes, showing that different patterns form when diffusion is present. A general study of these nonlinear partial differential equations, on different domains with different boundary conditions in particular, is to be pursued in future work, and I hope may inspire more specialised studies in any of the growing range of potential applications.

It is gratifying to see the works of pioneers like Thom and Turing still proving so fruitful and challenging, and even more-so, to present this work in memory of the enigmatic Soto, who was not only so instrumental to the success of singularity theory, but whose work on structural stability has both uplifted an entire field of study, and shown how singularity theory is of 'applied' and not just 'theoretical' importance — the obstacle Thom himself felt unable to surmount. Soto has been so influential to my own work, and in helping foster a wonderful community around dynamical systems in Brazil. The applied fields of singularities, bifurcations, and discontinuities, would be greatly impoverished without our charismatic Jorge Manuel Sotomayor Tello – our Soto.

A The $\mathcal{B}$-$\mathcal{G}$ determinants in long form

At the start of section 2 we defined certain determinants that are needed to define underlying catastrophes. The expressions there are in a rather compressed notation, so to assist the reader let me write (1)-(3) here in long form.

They are (giving the equations the same numbering as in the earlier text):

Firstly the $\mathcal{B}_i$ determinants,

$$\mathcal{B}_i = |\nabla(\mathcal{B}_{i-1}, f_2, ..., f_n)| = \det\begin{pmatrix} \frac{\partial \mathcal{B}_{i-1}}{\partial x_1} & \frac{\partial \mathcal{B}_{i-1}}{\partial x_2} & ... & \frac{\partial \mathcal{B}_{i-1}}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & ... & \frac{\partial f_2}{\partial x_n} \\ \vdots & & \ddots & \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & ... & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

with (1)

$$\mathcal{B}_1 = |\nabla(f_1, f_2, ..., f_n)| = \det\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & ... & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & ... & \frac{\partial f_2}{\partial x_n} \\ \vdots & & \ddots & \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & ... & \frac{\partial f_n}{\partial x_n} \end{pmatrix}.$$

If we use the helpful shorthand for the string $K(i) := k_1...k_{i-1}$, then we define the $\mathcal{G}_{i,K(i)}$ determinants as

$$\mathcal{G}_{i,K(i)} = \left|\Box\left(f_1, ..., f_n, \mathcal{B}_1, ..., \mathcal{B}_{i,K(i)}\right)\right|$$
$$= \det\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & ... & \frac{\partial f_1}{\partial x_n} & \frac{\partial f_1}{\partial \alpha_1} & ... & \frac{\partial f_1}{\partial \alpha_r} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & ... & \frac{\partial f_n}{\partial x_n} & \frac{\partial f_n}{\partial \alpha_1} & ... & \frac{\partial f_n}{\partial \alpha_r} \\ \frac{\partial \mathcal{B}_1}{\partial x_1} & ... & \frac{\partial \mathcal{B}_1}{\partial x_n} & \frac{\partial \mathcal{B}_1}{\partial \alpha_1} & ... & \frac{\partial \mathcal{B}_1}{\partial \alpha_r} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \mathcal{B}_{i,K(i)}}{\partial x_1} & ... & \frac{\partial \mathcal{B}_{i,K(i)}}{\partial x_n} & \frac{\partial \mathcal{B}_{i,K(i)}}{\partial \alpha_1} & ... & \frac{\partial \mathcal{B}_{i,K(i)}}{\partial \alpha_r} \end{pmatrix}, \quad (2)$$

in terms of the determinants $\mathcal{B}_{i,K(i)}$ defined as

$$\begin{aligned}\mathcal{B}_{i,K(i)} &= \left|\nabla\left(f_1, ..., f_{k_{i-1}-1}, \mathcal{B}_{i-1,K(i-1)}, f_{k_{i-1}+1}, ..., f_n\right)\right| \\ &= \det\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & ... & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial f_{h-1}}{\partial x_1} & ... & \frac{\partial f_{h-1}}{\partial x_n} \\ \frac{\partial \mathcal{B}_{i-1,K(i-1)}}{\partial x_1} & ... & \frac{\partial \mathcal{B}_{i-1,K(i-1)}}{\partial x_n} \\ \frac{\partial f_{h+1}}{\partial x_1} & ... & \frac{\partial f_{h+1}}{\partial x_n} \\ \vdots & & \ddots \\ \frac{\partial f_n}{\partial x_1} & ... & \frac{\partial f_n}{\partial x_n} \end{pmatrix},\end{aligned} \tag{3}$$

with $h = k_{i-1}$.

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