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Hidden Dynamics

The mathematics of switches, decisions, & other discontinuous behaviour

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A smooth sea never made a skillful sailor
– African proverb

Life has no smooth road for any of us;
and in the bracing atmosphere of a high aim
the very roughness stimulates the climber to steadier steps...
– William C. Doane

Artwork by Martin Williamson (http://www.cobbybrook.co.uk/) 2017

Chapter -1
Preface

Discontinuities are encountered when objects collide, when decisions are made, when switches are turned on or off, when light or sound refract as they pass between different media, when cells divide or neurons are activated, ... examples are to be found throughout the modern applications of dynamical systems theory.

Mathematicians and physicists have long known about the importance of discontinuities. Studying light caustics (the intense peaks that create rainbows or the bright ripples in sunlit swimming pools), George Gabriel Stokes lamented to his fiancée in a letter from 1857:

... sitting up til 3 o'clock in the morning
... I almost made myself ill, I could not get over it
... the discontinuity of arbitrary constants.

Discontinuities are not a welcome feature in dynamical or differential equations, because they introduce indeterminacy, the possibility of one problem having many possible solutions, many possible behaviours. How interesting it is then to consider the thoughts of the influential engineer Ove Arup:

Engineering is not a science ... its problems are under-defined,
there are many solutions, good, bad, or indifferent.
The art is ... to arrive at a good solution.

For Arup, "*science studies particular events to find general laws*". Many mathematical scientists would agree that the goal is to achieve generality, and banish indeterminacy. But why should the two be mutually exclusive?

Unlocking the potential of discontinuities requires tackling these issues of determinacy and generality. While accepting that some parts of the world lie beyond precise expression, discontinuities nonetheless give us a way to express them, to approximate that which cannot be approximated by standard 'well-posed' equations, and to explore a world where certain things will ever remain hidden from view.

With these ideas in mind, this book attempts to ready the field of nonsmooth dynamics for turning to a wider range of applications, simultaneously moving beyond the traditional scope of, and bringing our subject closer into line with, the traditional theory of differentiable dynamical systems.

At a discontinuity we lose access to some of the most powerful theorems of dynamical systems, and it has long been the task of nonsmooth dynamical theory to redress this. Progress has been impressive in some areas, limited in others. We suggest here that much of what has gone before constitutes a *linear* approach to discontinuities, and here we lay the foundations for a *nonlinear* theory. Making use of advances in nonlinearity and asymptotics, once we can extend elementary methods such as linearization and stability analysis to nonsmooth systems, discontinuities stop being objects of nuisance, and start becoming versatile tools to apply to modeling the real world.

Several examples of applications are studied towards the end of the book, and many more could have been included. Interest in piecewise-smooth systems has been spreading across scientific and engineering disciplines because they offer reliable models of all manner of abrupt switching processes. Our aim is to set out in this book the basic methods required to gain an in-depth understanding of discontinuities in dynamics, in whatever form they arise.

In this book, a discontinuity is *blown up* into a *switching layer*, inside which *switching multipliers* evolve infinitely fast across the discontinuity. Several concepts may be at least partly familiar in other areas of mathematics, in particular algebraic geometry, boundary layers, singularity theory, perturbation theory, and multiple timescales. The terminology used here does not exactly correspond to the usage in those fields, and attempting to refer to or resolve all of the clashes in nomenclature would not make for an easier read. Moreover we do not use the concepts themselves in strictly the same way. For example, we use the idea of an infinitesimal ε-width of a discontinuity that we can manipulate algebraically, but we are interested solely in the limit $\varepsilon = 0$. This proves to be a sufficiently rich problem, and though it raises the question of what happens when we perturb to $\varepsilon > 0$, that is left for future work. As we discuss in chapter 1 and chapter 12, moreso than in any smooth system, the perturbation of a discontinuity is a many faceted problem.

Thus work builds on the pioneering efforts particularly of Aleksei Fedorovich Filippov, Vadim I. Utkin, Marco Antonio Teixeira, and Thomas I. Seidman. I have been lucky to meet and work with all but the first of these, and much in the spirit of modern science, they represent the truly international and interdisciplinary endeavour of what was for too long a niche field of study.

In essence, the framework introduced in this book seeks to explore Filippov's world more explicitly, to make non-uniqueness itself a useful modeling tool in dynamics. Sitting somewhere between deterministic dynamics and stochastic dynamics, nonsmooth dynamics offers a third way: systems that are only piecewise-defined, rendering them *almost everywhere* deterministic.

Chapter 0
Chapter outline

The book is roughly split into 3 parts: introductory material in chapters 1 to 2, fundamental concepts at the level of the student or non-expert in chapters 3 to 6 and chapter 14, and advanced topics in chapters 7 to 13.

Chapter 1 is almost a standalone and informal essay, surveying the reasons why discontinuities occur, the forms they take, why they matter, and how imperfect our knowledge of them is. The chapter is intended to provoke thought and discussion, not to be detailed reference on the many theoretical and applied concepts it touches on.

Chapter 2 is a standalone "lecture" style outline, a crash course on the topic and a taster of the main concepts that will be developed in the book.

Chapter 3 contains the complete foundation for everything that follows, the formalities for how we define piecewise smooth systems in a solvable way. This chapter contains the elements necessary for the eager researcher to rediscover for themselves the contents of the remainder of the book and beyond.

Chapter 4 sets out the basic themes that dominate piecewise-smooth dynamics, the kinds of orbits, the key singularities, and the concepts of stability and bifurcation theory.

Chapter 5 defines a general prototype expression for piecewise-smooth vector fields in the form of a series expansion.

Chapter 6 describes the basic forms of contact between a flow and a discontinuity threshold.

Chapter 7 contains the most important new theoretical elements of the book, setting out the analytical methods required to understand piecewise smooth systems.

Chapter 8 takes a step back, applying the previous chapters in the more standard setting of linear switching.

Chapter 9 begins the leap forward into nonlinear switching, revealing some of the novel phenomena of piecewise smooth systems.

Chapter 10 focusses on the most extreme consequences of discontinuity, via determinacy breaking and loss of uniqueness.

Chapter 11 tackles how we understand large-scale behaviour, with new notions of global dynamics and associated bifurcations.

Chapter 12 asks how robust everything that has come before is. We consider how our mathematical framework can be interpreted in a practical setting, and what happens to it in the face of non-ideal perturbations.

Chapter 13 visits an old friend and long term obsession of piecewise smooth systems, the two-fold singularity.

Chapter 14 is a series of case studies applying the foregoing analysis to 'real world' models.

Exercises are provided at the back of the book to further facilitate a more in-depth reading or lecture course.

How to use this book

This book will look rather different to other works in the area. In chapter 1 we start from a tour of some less quoted, wideranging, but fundamental, examples of how discontinuity arises. Chapter 3 presents the formalism for studying nonsmooth dynamics that forms the foundation for everything that follows, and should be the starting point for any course. It is quite possible to jump from there to chapter 12 to focus on the application and robustness of the formalism.

A proper understanding of the dynamics of nonsmooth system, or a course in it, should progress through chapter 4 to 11, and I would suggest focussing on (and indeed extending) the analytical methods in chapter 7. The great peculiarities of nonsmooth systems begin to be revealed in chapter 9 and chapter 10, and there are numerous examples therein to explore and build on. More in-depth applications are given in the form of case studies in chapter 14. Exercises provided at the back of the book provide further insight into the various examples and theorems explored, chapter by chapter.

Prerequisites. In reading this book it will be helpful to have a grounding in (though we give elementary introductions where possible): single and multi-variable calculus, Taylor series, ordinary differential equations and elementary dynamical systems, some linear algebra (eigenvectors etc.), and a little introductory (highschool) physics. Applications will be explained with background where they are discussed.

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