Integral Curves of a Vector Field with a Fractal Discontinuity

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Nonsmooth systems are typically studied with smooth or piecewise-smooth boundaries between smooth vector fields, especially with linear or hyper-planar boundaries. What happens when there is a boundary that is not as simple, for example a fractal? Can a solution to such a system slide or "chatter" along this boundary? It turns out that the dynamics is intricate and fascinating, and yet contained within A. F. Filippov's theory (as promised in V. I. Utkin's article [**2**] in this volume).

As motivation, take a simple two-dimensional system with a discontinuity boundary formed of the Koch curve of height ε, for small ε. First consider the vector fields pointing horizontally above the surface and vertically below.

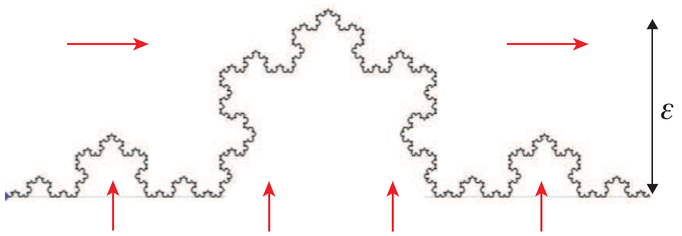

Figure 1. The two vector fields separated by the Koch snowflake fractal boundary.

The boundary of the Koch curve has infinite length, and indeed between any two points on the curve the length is infinite. However, the time spend on smaller and smaller segments of the surface is ever decreasing. To find a solution to the problem

$$(\dot{x}, \dot{y}) = \begin{cases} (\beta, 0) & \text{if} \quad h > 0\,, \\ (0, 1) & \text{if} \quad h < 0\,, \end{cases} \tag{1}$$

for some β, from, say, an inital condition at the left extreme of the curve as shown, requires a recursive calculation.

The solution can only move up and to the right, via either vector field (with the vector field so oriented, sliding does not occur) along the curve. This implies that the solution eventually reaches the highest peak of the switching surface. Having travelled only horizontally or vertically, the total distance travelled must be $1/2 + \sqrt{3}/3$. A similar argument can be made to show the solution must reach the smaller peak shown in Figure 2 below. From that peak, the solution will move right until it reaches the fractal boundary again, and then upward until it again hits the fractal boundary shown below. From there the fractal is a copy of the previous step, so we can recursively iterate this path, scaling by $1/3$ each time, until it reaches the largest peak.

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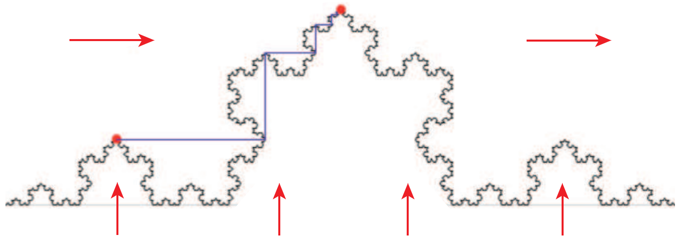

FIGURE 2. The solution between the largest peak and second largest peak (dotted), iteratively moving right and upward.

The path through the first third of the path is a replica of the path we just drew, since the fractal contains a scaled copy of itself. Recursively filling in this path we can generate the full integral curve, with a total length of $1/2 + \sqrt{3}/3$. The integral curve is itself a fractal, but one of finite length.

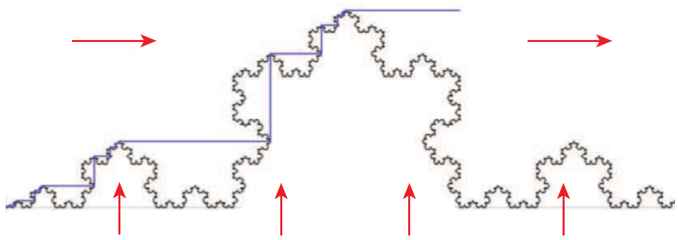

FIGURE 3. The full solution with initial condition at the left extreme of the curve.

What then happens if both vector fields impinge on the switching surface, e.g.

$$(\dot{x}, \dot{y}) = \begin{cases} (\beta, -\beta) & \text{if} \quad y > 0\,, \\ (0, 1) & \text{if} \quad y < 0\,. \end{cases} \tag{2}$$

for $\beta > 0$. The motion is sketched approximately in Figure 4.

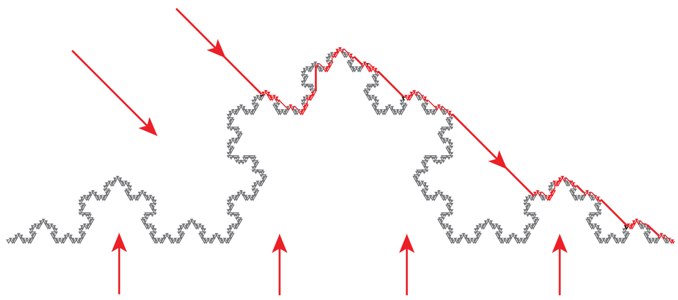

FIGURE 4. Two vector field convering on a Koch curve switching surface, and an integral solution.

Whatever form the motion takes, denoting the upper and lower vector fields as (f^+, g^+) and (f^-, g^-), the speed of travel along the x-direction must be

$$\dot{x} = \lambda f^+ + (1 - \lambda) f^-\,, \tag{3}$$

where the motion consists only of a proportion of time λ in f^+ and $1-\lambda$ in f^-. Even if the motion consists of a proportion of time $\mu^\pm$ in $f^\pm$, and the remaining propotion of time $1-\mu^+-\mu^-$ sliding along the surface with horizontal speed f^s, then

$$\dot{x} = \mu^+ f^+ + \mu^- f^- + (1-\mu^+-\mu^-)f^s \tag{4}$$

but since $f^s = \lambda f^+ + (1-\lambda)f^-$ (by Filippov's method [**1**]),

$$\begin{aligned} \dot{x} &= \mu^+ f^+ + \mu^- f^- + (1-\mu^+-\mu^-)\{\lambda f^+ + (1-\lambda)f^-\} \\ &= \nu f^+ + (1-\nu)f^- \end{aligned} \tag{5}$$

where $\nu = \mu^+ + (1-\mu^+-\mu^-)$. Solving for a value of ν that gives motion along the switching surface, approximating the average vertical motion as $\Delta y = \mathcal{O}(\varepsilon) \approx 0$, we find $\nu = g^-/(g^- - g^+)$ hence $\dot{x} = (g^- f^+ - g^+ f^-)/(g^- - g^+)$, in either case (i.e. with or without actual sliding along the surface), consistent with ideal Filippov sliding on $y = 0$.

A simple example will establish the principle behind this. Take (2) where $h = 0$ is some complex threshold. Let us consider first a basic piecewise linear surface, letting the switching surface be comprised of a sawtooth of angle α. As depicted in the figure below, depending on whether $\alpha < \pi/4$ or $\alpha > \pi/4$, solutions slide along the sawtooth once they impact it, or slide and detach repeatedly.

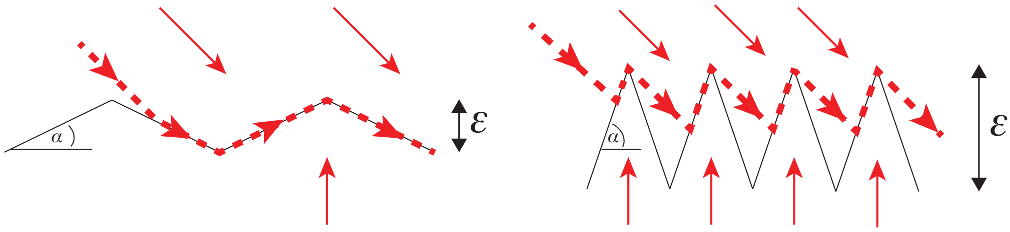

FIGURE 5. Motion along a sawtooth switching surface if shallower (left) than the upper vector field, or steeper (right) so that stick-slip occurs.

For $\alpha < \pi/4$, the vector fields are always pointing into the surface, so sliding will occur everywhere. The sawtooth inclines have normal vectors $(\pm\sin\alpha, \cos\alpha)$, so the sliding condition for the vector fields above is

$$\begin{pmatrix} \pm\sin\alpha \\ \cos\alpha \end{pmatrix} \cdot \left\{ \lambda \begin{pmatrix} \beta \\ -\beta \end{pmatrix} + (1-\lambda) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \quad \Rightarrow \quad \lambda = \frac{\beta^{-1}}{1+\beta^{-1} \mp \tan\alpha}\,, \tag{6}$$

and the average speed over the two inclines is

$$\langle \dot{x} \rangle = \frac{2}{\dot{x}_+^{-1} + \dot{x}_-^{-1}} = \frac{2}{2(1+\beta^{-1})} = \frac{1}{1+\beta^{-1}}\,. \tag{7}$$

For $\alpha > \pi/4$, firstly, if the distance between peaks is 1, and the distance travelled in the upper vector field before hitting the next incline is μ, then $\tan\alpha = \frac{\mu}{1-\mu}$, hence $\mu = 1/(1+\cot\alpha)$. Then the speed, averaging over motion through the upper vector field and sliding on the upward incline (using the sliding vector field above) is

$$\begin{aligned} \langle \dot{x} \rangle &= \frac{1}{\frac{\mu}{\dot{x}_+} + \frac{1-\mu}{\dot{x}_-}} = \frac{1}{\mu\beta^{-1} + (1-\mu)(1+\beta^{-1}+\tan\alpha)} \\ &= \frac{1+\tan\alpha}{\beta^{-1}\tan\alpha + 1 + \beta^{-1} + \tan\alpha} = \frac{1}{1+\beta^{-1}}\,. \end{aligned} \tag{8}$$

Let us compare these two results to the sliding vector field for a flat surface $y = 0$. Solving $\dot{y} = \lambda(-\beta) + (1-\lambda) = 0$ implies $\lambda = 1/(1+\beta)$ giving $\dot{x} = 1/(1+\beta^{-1})$. Hence the form of the motion does not change the sliding vector field approximation – Filippov's method holds.

These results are independent of the size of the sawtooth pattern. Likewise if we calulate the portions of motion through the upper vector field, lower vector field, and sliding vector field along the Koch curve, regardless of the fractal structure of the path, ultimately the distance travelled is finite and the speed of motion averages out to Filippov's sliding vector field.

The result extends, of course, to many other structures where motion entirely consists of motion through either the upper or lower vector field, or sliding according to the convex combination along the boundary between them. The switching surface might consist of a layer tiled with regions on which one or other vector field apply, with sliding and crossing regions on their edges, again perhaps of a fractal structure.

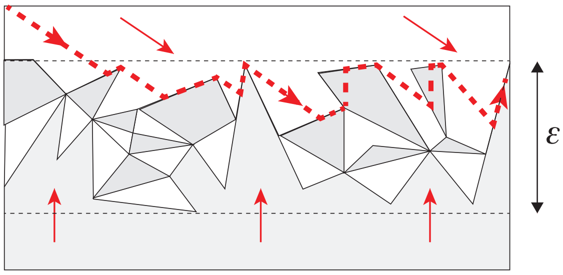

FIGURE 6. Along a tiled switching surface of thickness $\varepsilon \to 0$, for the vector fields above, solutions will slide at a speed $\langle \dot{x} \rangle = 1/(1+\beta^{-1})+\mathcal{O}(\varepsilon)$.

While an entertaining problem, this has the more serious aim of clarifying the nature of switching surfaces to which the sliding concept applies. Filippov's convex combination methodology has very wide applications, and makes less assumptions about the nature of the switches surface than might be thought.

Finding the integral curves themselves is not a trivial problem, and although we have argued above that the sliding mode must be as in Filippov's theory, the explicit calulation above should be extended to demonstrate how the limit along a fractal surface tends to the convex combination result. It is possible that as yet unforseen dynamical issues may arise with more interesting surfaces, for example in the case of two switches, a fractal surface can undoubtedly be expected to have less trivial consequences.

References

[1] A. F. Filippov. *Differential Equations with Discontinuous Righthand Sides*. Kluwer Academic Publ. Dortrecht, 1988 (Russian 1985).

[2] V. I. Utkin *Comments for the continuation method by A. F. Filippov for discontinuous systems*. Trends in Mathematics: Research Perspectives CRM Barcelona (Springer), 2016.

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