11th International Conference on Engineering Vibration
Ljubljana, Slovenia, 7–10 September 2015

HIDDEN DYNAMICS OF DRY-FRICTION OSCILLATORS

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Keywords: nonsmooth, switching, dynamics, sliding, sticking, discontinuity, singularity, bifurcation

Abstract.

The theory of nonsmooth dynamics describes how discontinuities in dynamical laws, such as those caused by friction, impacts, shape buckling, or mechanical relays, affect the deterministic and otherwise smooth behaviour of a mechanical system. In the geometric theory of dynamics, a system 'flows' through a phase space, sometimes encountering limit sets, bifurcations, or chaos. The phase space of a nonsmooth flow is also permeated by switching thresholds. The flow can kink as it crosses the threshold, but it can also stick to the threshold, corresponding to frictional sticking, and resulting phenomena like stick-slip oscillations have been a fruitful area of study for mechanical modeling. The interaction of multiple objects undergoing coupled stick-slip oscillations have not been studied from the same point of view, however, because, perhaps surprisingly, the continuous time dynamical methods for multiple sticking events were derived only recently.

We summarize those methods here, including the extension of Filippov's methods to multiple switches, and the introduction of hidden dynamics inside a discontinuity. We show the implications these have for a series of coupled dry friction oscillators, giving insight into complex self-sustained oscillations. We derive the basic mechanisms of entry and exit from single or multiple sticking modes (i.e. sticking of multiple oscillators), which include both deterministic and determinacy-breaking exit points. Both of these kinds of exit point can cluster in higher dimensional systems, and both lead to complexity of behaviour in the form of robust, repeatable, but unpredictable behaviour. The study of exit points reveals how large scale unpredictability, with no obvious global structure, nevertheless has local origins in the form of local sensitivity to initial conditions at exit point singularities.

1 Introduction

One might not expect that much remains to be learned about the elementary mechanics underlying dry friction oscillators, especially when assuming the simplest Coulomb-type (step function) contact force. In this paper we begin showing how advances in the mathematical theory of nonsmooth dynamics alter our view of how complex mechanical phenomena emerge from simple underlying geometry, and highlight some of the techniques that can be used to study them.

The systems we will consider relate a set of displacements $\mathbf{x} = (x_1, x_2, ..., x_n)$, and their time derivatives, by a relation of the general form

$$\ddot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \dot{\mathbf{x}}; \boldsymbol{\lambda}) \;, \tag{1}$$

expressible as a first order system in terms of $\mathbf{x}$ and $\mathbf{y}$ if we let $\mathbf{y} = \dot{\mathbf{x}}$. A vector of parameters $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, ..., \lambda_m)$ will typically represent contact forces which obey

$$\lambda_i \in \begin{cases} \operatorname{sign}(h_i(\mathbf{x})) & \text{if} \quad h_i(\mathbf{x}) \neq 0 \;, \\ [-1, +1] & \text{if} \quad h_i(\mathbf{x}) = 0 \;, \end{cases} \tag{2}$$

for $i = 1, 2, ...m$, in terms of smooth scalar functions $h_1, h_2, ...\, h_m$. On each surface $h_i(\mathbf{x}) = 0$, if there exists a solution to the inclusion (1)-(2) such that $\dot{h}_i(\mathbf{x}) = 0$, then the system has trajectories that *stick to* (or in the dynamics system parlence 'slide along') the surface $h_i = 0$ for some interval of time. Likewise if $\dot{h}_i(\mathbf{x}) = 0$ on $h_i(\mathbf{x}) = 0$ can be solved for a subset of $i \in \{1, 2, ..., m\}$, trajectories exist that stick to the intersection of the surfaces $h_i = 0$. The general principles of how to determine the existence and dynamics of sticking modes can be found in [8], but for the mechanical oscillator they are particularly simple.

In the following we shall consider one, two, and then multiple oscillators, coupled via springs and dampers, situated on a surface moving with constant speed and subject to Coulomb dry friction. The functions h_i will become the slipping speed of the i^{th} oscillator relative to the surface, positive and negative h_i denoting right and leftward motion, respectively. We will look at the mechanics by which oscillators are released from stick ($h_i = 0$) to slip ($h_i \neq 0$), observed as singularities in the phase space of $\mathbf{x}$ and $\dot{\mathbf{x}}$, and show how this leads to either *deterministic* or *determinacy-breaking* exits from sticking. We then discuss the way clusters of exit events tend to appear, creating a likelihood that multiple oscillators will exit simultaneously, or in close succession, from stick to slip. The effect of this geometry on large numbers of oscillators is then studied in numerical experiments, revealing short term unpredictability against a background of long term self-organisation, which reveals itself in the statistics of stick-slip cascades.

2 Slip and Stick for a single oscillator

We begin with the simple dry friction oscillator, reviewing certain basic insights that we shall extend to many oscillators below. A block rests on a surface that moves with constant speed v. The block is attached to a fixed apparatus by a spring with stiffness κ and extension z, and by a dashpot with damping coefficient ρ. The block's mass can be scaled out of the equations, resulting in

$$\ddot{z} + \rho\dot{z} + \kappa z + \mu\,(\dot{z} - v) = 0 \;, \tag{3}$$

where $\mu(\dot{z} - v)$ is a function describing the coefficient of friction between the block and the surface, satisfying

$$\mu(u) \in \tilde{\mu} \begin{cases} \operatorname{sign}(u) & \text{if} \quad u \neq 0 \;, \\ [-1, +1] & \text{if} \quad u = 0 \;. \end{cases} \tag{4}$$

The constant $\tilde{\mu}$ denotes the coefficient of kinetic friction. We can form a first order ordinary differential equation by letting $\dot{z} = y + v$, where y is the block's velocity relative to the moving surface. This and similar systems have been extensively studied from analytical, numerical, and experimental viewpoints, see e.g. [4, 7, 12, 13], especially with regard to periodic or chaotic dynamics when a periodic forcing is added to (3).

The block's motion is illustrated in phase space in fig. 1, as a primer for the more complicated situations to follow. The block switches its direction of slip as $\dot{z} - v$ passes through zero, but only if the spring force $(-\kappa z)$ is great enough that $(\rho v + \kappa z)^2 > \tilde{\mu}^2$, otherwise the block and the surface stick together, seen as the trajectory traveling along $\dot{z} = v$ in fig. 1. During the sticking phase we can find the resistance force provided by considering μ to be a variable in (3), and finding its value by solving

$$\ddot{z} + \rho\dot{z} + \kappa z + \mu = 0 \;, \quad \dot{z} = v \;, \quad \ddot{z} = 0 \qquad \Rightarrow \qquad \mu = -\rho v - \kappa z \;. \tag{5}$$

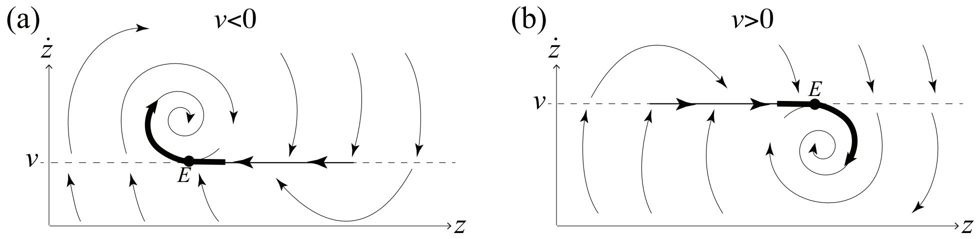

Figure 1: The phase space of the dry friction oscillator for $v < 0$ and $v > 0$. Slipping motion to the right or left correspond to relative speeds $\dot{z} > v$ or $\dot{z} < v$ respectively. Crossing between slip directions takes place where the line $\dot{z} = v$ is dashed, while sticking occurs between the two points where the flows fold towards or away from the line. A fold where the flow curves away creates an exit point (marked E), where the oscillator goes from stick to right slip in (i) or from stick to left slip in (ii), and the exiting trajectory is shown as a bold curve.

The boundary between the two cases (slip-slip transition or slip-stick transition) is delimited by folds in the slipping flow at $z = (\mp\tilde{\mu} - \rho v)/\kappa$, where the flow is quadratically tangent to the discontinuity. Whether the flow turns towards or away from the discontinuity surface at these points is determined by the curvature term $\dddot{z} = -\rho\ddot{z} - \kappa\dot{z} = -\kappa v$, which implies that the flow curves upwards at both folds if $v < 0$ (downwards if $v > 0$ but we mainly consider the former here). Places where the flow curves away from the discontinuity create *exit points* from sticking (the point E in the figure), and will take on increasing importance in what we consider below.

In systems of multiple oscillators studied in the next two sections, it is possible for exit points like those in fig. 1(i-ii) to form on both sides of the discontinuity simultaneously. This causes an ambiguity in which side of the discontinuity a trajectory will exit from sticking. The general ambiguity has been much studied in the last two decades (see [3] and references therein), but seems not to arise easily in low dimensional mechanical systems. Below we shall see that ambiguous exits play a more significant role as more oscillators are added.

3 Slip and Stick for a pair of oscillators

Consider now a pair of blocks each independently of the form (3) (labelled by subscripts 1 and 2), and coupled by an additional spring with stiffness κ_{12}. We have

$$\begin{aligned} &\ddot{z}_1 + \rho_1\dot{z}_1 + \kappa_1 z_1 + \kappa_{12}(z_1 - z_2) + \mu_1\,(y_1 - v) = 0 \;, \\ &\ddot{z}_2 + \rho_2\dot{z}_2 + \kappa_2 z_2 + \kappa_{12}(z_2 - z_1) + \mu_2\,(y_2 - v) = 0 \;. \end{aligned} \tag{6}$$

Again v is the speed of the surface on which the blocks rest. Such systems have been much studied for their periodic or chaotic dynamics under periodic forcing, see e.g. [1], but, as for the single oscillator, we are more interested in local properties and omit any external forcing. The coupled system dynamics occupies the phase space of $(z_1, z_2, \dot{z}_1, \dot{z}_2)$, with discontinuity at the two surfaces $z_1 = v$ and $z_2 = v$, where the two blocks switch between left and right slip.

Consider the discontinuity surface $y_1 = v$ first. Sticking occurs for displacements z_1, z_2, satisfying $(\rho_1 v + \kappa_1 z_1 + \kappa_{12}(z_2 - z_1))^2 < \tilde{\mu}_1^2$, and elsewhere on $y_1 = v$ the first block flips between left and right slip without sticking. The two cases are separated, as in section 2, by folds of the slipping flows, which occur on planes $(\kappa_1 - \kappa_{12})z_1 + \kappa_{12}z_2 = \mp\tilde{\mu}_1 - \rho_1 v$. Whether the flow turns towards or away from the discontinuity is determined by the curvature term $\dddot{z}_1 = (\kappa_{12} - \kappa_1)v - \kappa_{12}\dot{z}_2$, whose sign is dependent on the velocity $\dot{z}_2$ of the second block, and on the relative sizes of κ_1 and κ_{12}. Clearly the quantity $\dddot{z}_1$ itself can vanish, creating a cusp or higher order tangency between the flow and the discontinuity, see e.g. [5, 14].

We can perform similar analysis for the second block to find that, on the discontinuity surface $y_2 = v$, sticking occurs where $(\rho_2 v + \kappa_2 z_1 - \kappa_{12}(z_2 - z_1))^2 < \tilde{\mu}_2^2$, folds occur on planes $(\kappa_2 - \kappa_{12})z_2 + \kappa_{12}z_1 = \mp\tilde{\mu}_2 - \rho_2 v$, where the curvature term $\dddot{z}_2 = (\kappa_{12} - \kappa_2)v - \kappa_{12}\dot{z}_1$ decides the existence of exit points.

Sticking of both blocks occurs where both sticking conditions hold, on a region taking the form of a parallelogram in the (z_1, z_2) space on $\dot{z}_1 = \dot{z}_2 = v$ (bottom-left in fig. 2),

$$\begin{array}{c} -\tilde{\mu}_1 < \rho_1 v + (\kappa_1 - \kappa_{12},\ \kappa_{12}) \cdot (z_1,\ z_2) < +\tilde{\mu}_1\ , \\ -\tilde{\mu}_2 < \rho_2 v + (\kappa_{12},\ \kappa_2 - \kappa_{12}) \cdot (z_1,\ z_2) < +\tilde{\mu}_2\ . \end{array} \tag{7}$$

The edges of this region are the two fold planes. As a trajectory leaves the parallelogram (7) at one of these edges, one of the blocks begins to slip, and may be closely followed by the second block beginning to slip. Figure 2(a) depicts a bold trajectory where block 1 and then block 2 slip in close succession.

Our greater interest now resides in the corners of the parallelogram, where exit points coincide when both blocks may exit from sticking contact simultaneously. These are two-fold singularities of the kind described in [10], where there is a fold both of the first and second oscillator's flow. Solving both of the fold conditions to find the corner gives the isolated points

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = -\tfrac{\rho_1 v \pm \tilde{\mu}_1}{K} \begin{pmatrix} \kappa_{12} - \kappa_1 \\ \kappa_{12} \end{pmatrix} - \tfrac{\rho_2 v \pm \tilde{\mu}_2}{K} \begin{pmatrix} \kappa_{12} \\ \kappa_{12} - \kappa_2 \end{pmatrix}\ , \tag{8}$$

writing $K = \kappa_1\kappa_2 - \kappa_{12}(\kappa_1 + \kappa_2)$. The different combinations of $\pm\tilde{\mu}_1$ and $\pm\tilde{\mu}_2$ give four such points. The curvature in the $\dot{z}_1$ and $\dot{z}_2$ directions takes the same value at each point, namely

$$\dddot{z}_1 = -\kappa_1 v\ , \qquad \dddot{z}_2 = -\kappa_2 v\ , \tag{9}$$

meaning for $v < 0$ the flow curves always in the position direction with respect to the $\dot{z}_1$ and $\dot{z}_2$ directions, and thus towards the discontinuity from below, and away from it from above. Hence the point given by $\{+\tilde{\mu}_1, +\tilde{\mu}_2\}$ is a double exit point, where both blocks are carried away from sticking by the flow, and therefore both begin slipping simultaneously. The points given by $\{+\tilde{\mu}_1, -\tilde{\mu}_2\}$ and $\{-\tilde{\mu}_1, +\tilde{\mu}_2\}$ are illustrated in fig. 2 (b) and (c) respectively. The dynamics in each scenario are described in the caption, and reveal that only (b) forms an exit point, in which both blocks simultaneously cease sticking.

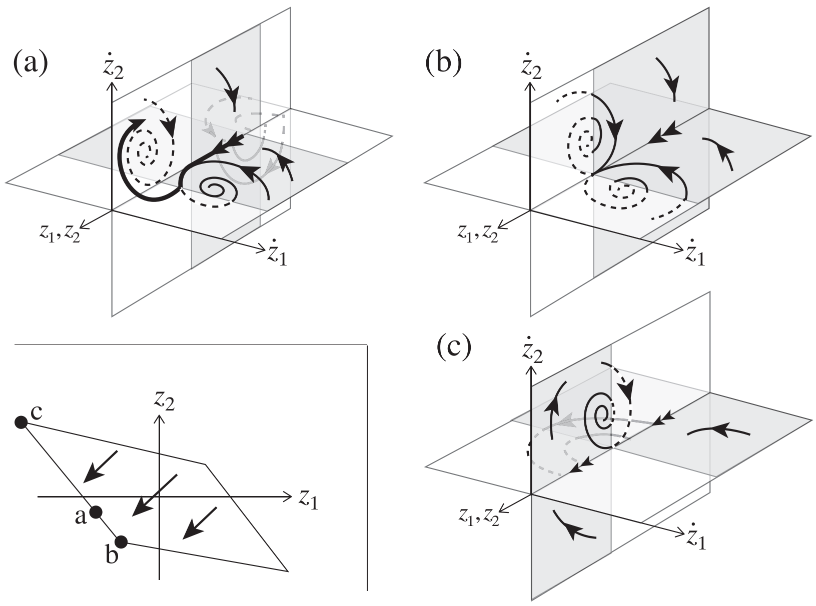

Figure 2: A representation of dynamics in the four dimensional phase space of $(z_1, z_2, \dot z_1, \dot z_2)$, showing the discontinuity surfaces $\dot z_1 = v$ and $\dot z_2 = v$, on which sticking occurs in the shaded regions. Illustrating: (a) a typical case, showing the flow on the discontinuity surfaces when sticking occurs (dotted curves show the'virtual' flow, indicating the flow off the surfaces where there is no sticking); (b) a visible two-fold forms a double exit point, a boundary where both oscillators cease sticking, and since the exit takes place into a region where there is no sticking the flow is well-determined; (c) a two-fold of mixed visibility has a well-defined trajectory emanating from it, but the flow in the (z_1, z_2) plane reveals it is not an exit point as the flow cannot enter the point (c) along the intersection. This is a sketch generated from simulations for constants $\rho_1 = \rho_2 = 0.1$, $\kappa_1 = 1.1$, $\kappa_2 = 1.6$, $\kappa_{12} = 0.5$, $\tilde\mu_1 = \tilde\mu_2 = 1$, $v = -1$. The different points are shown in (z_1, z_2) space (bottom-left) as the corners of the sticking parallelogram (7), whose edges are the fold sets, and on which the flow during sticking obeys $(\dot z_1, \dot z_2) = (v, v)$.

4 Slip and Stick for multiple oscillators

We now generalize the above system to any number of oscillators, constrained to move along a line, but to avoid being overly prescriptive, we allow them to be coupled by an arbitrary arrangement of springs. The system we shall study here extends (3) and (6) to n blocks by writing

$$\ddot z_i + M^\rho_{ij}\dot z_j + M^\kappa_{ij} z_j + \mu_i (y_i - v) = 0\ , \qquad i = 1, ..., n, \tag{10}$$

where M^ρ and M^κ are $n \times n$ square matrices, and we sum over the index $j = 1, ..., n$. The matrix of damping coefficients M^ρ is diagonal, the matrix of spring stiffness coefficients M^κ has a diagonal part (of individual springs) and an antisymmetric part (of spring couplings).

Systems of multiple coupled oscillators similar to this have again been much studied, but the continuous time approaches used for the low dimensional systems as above tend to be abandoned in favour of discrete methods. Such models are now a huge topic of network complexity science, building on earthquake models such as [2], and typically modelled as discrete time algorithmic models (where 'stick' and 'slip' are discrete modes). In the light of the continuous time dynamics above, and exit points in particular, we shall now re-consider what the continuous time dynamical theory implies for the behaviour of a high dimensional system.

In section 2 the system had two fold points which bounded the sticking dynamics. In section 3 the system had four lines of folds which intersected in pairs (bottom-left in fig. 2), near which there is an increased chance of one exit (one block beginning to slip) being followed by another (the second block beginning to slip). As more oscillators are introduced, say a total of

n oscillators, we obtain $2n$ folds. If r oscillators are sticking, the dynamics occupies a $2n-r$ dimensional phase space, upon which k folds may intersect where $n+k \leq 2n-r$, and there will be up to 2^k such points. Thus there exist up to 2^k points where up to $k \leq n$ of $r \leq k$ sticking oscillators may begin slipping simultaneously, nearby by which cascades of up to 2^k oscillators may be seen beginning to slip, one after another in rapid succession.

Obviously, displaying simulations of systems of multiple oscillators in phase space will be of limited use, and already in section 3 we have been able to give only schematic representations of the phase space dynamics for two blocks. From that analysis, however, the central feature of interest is the codimension of the motion, equivalent to the number of intersecting discontinuity surfaces upon which sticking motion occurs, or, in physical terms, simply the number of blocks sticking at a given time. We label this the *sticking codimension* n_{stick}. Exit points are then easily observed as incremental decreases in the sticking codimension. If clusters of exit points occur as expected, they will be seen as cascades of decreasing sticking codimension n_{stick}.

The following plots are taken from a simulation of the system (10) with $v = -0.2$, with all $\tilde{\mu}_i = 1$, and with initial conditions $z_i = -0.7$, $\dot{z}_i = -0.4$, for all oscillators. The damping constants are taken as random values in the range $M^{\rho}_{ij} \in [\rho, 2\rho]$ with $\rho = 3.5$, the spring constants as random values in the range $M^{\kappa}_{ij} \in [-\kappa, +\kappa]$ with $\kappa = 0.2$. These various details have little impact on the qualitative outcomes of the simulations, the key parameters which provide the behaviour below being v, ρ, and κ.

Figure 3(i) shows the sticking codimension for a system of 20 oscillators. Four oscillators stick almost immediately, followed by a succession of exit points through which all slip at around $t \approx 24$, and after a short return to sticking all oscillators eventually slip. The behaviour becomes more interesting as we increase the dimension. For 100 oscillators, in fig. 3(ii), numerous cascades of exit points, and countering returns to stick, occur over a time period of around 100 or so increments (but after $t \gtrsim 150$ all oscillators are eventually found to slip). For 200 oscillators, in fig. 3(iii), these complex stick-slip transitions become self-sustaining, with cascades of exit points and collapse back to sticking mediating each other over long times.

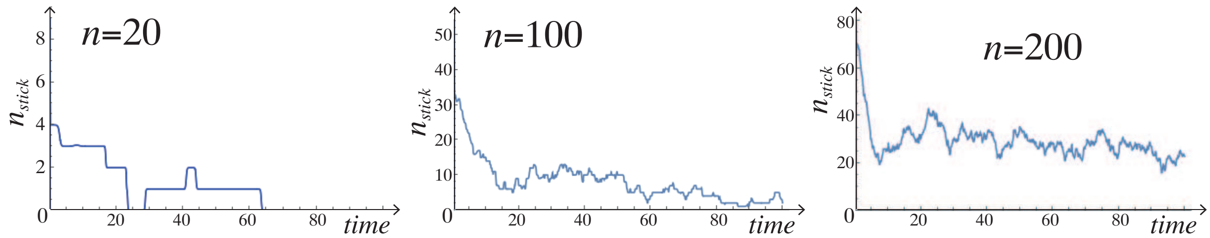

Figure 3: Plots of the sticking codimension n_{stick} for systems of n oscillators, as described in the text.

We shall look closer into this long-term behaviour below. First, as discussed in section 3, a system becomes very sensitive around exit points and should be sensitive to modeling assumptions. In fig. 4 we investigate the system's robustness, first to perturbations of the switching model and then to perturbations of the mechanical constants. An important simplification we have made in the simulations here is to approximate each sign function $\mu_i(u)$ by a sigmoid $\mu(u) = \phi(u/\epsilon)$, in fact we take $\phi \equiv \tanh$ and $\epsilon = 0.03$, and this permits us to simulate using a standard ODE package (in this case the Mathematica® routine NDSolve). The precise choice of sigmoid does not matter typically (see e.g. [9, 11]), but as noted in [6] can create novel behaviours near exit points. In fig. 4(i)-(ii) each switching function $\mu_i(u)$ is replaced by a sigmoid

$\mu(u) = \tanh(u/r_i\epsilon)$, each with a different stiffness $r_i\epsilon$ where the r_i's are a random numbers between 0 and 1. The result of three such simulations for 20 and 200 oscillators are shown. As expected if exit points with sensitivity to initial conditions are involved, the short-time dynamics is sensitive to these small changes in the modeling stiffness. The long time behaviour, however, is largely unaffected, even exhibiting similar large scale structures in the sustained oscillations of 200 oscillators. That is, we observe short-term sensitivity but long-term robustness, despite the complexity of the long-term dynamics. In fig. 4(iii)-(iv) we probe the robustness of

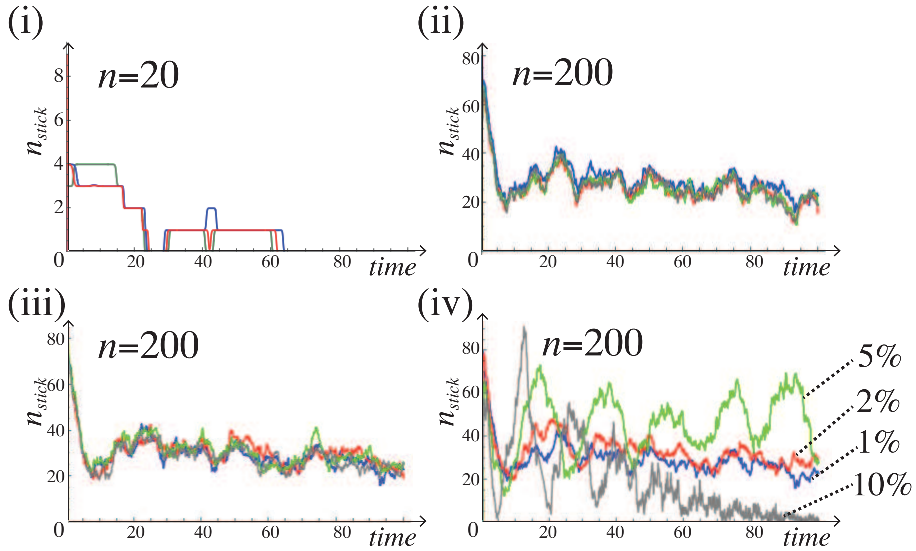

Figure 4: Plots of the sticking codimension n_{stick} for systems of n oscillators, simulated: (i-ii) for random perturbations of the stiffness parameter $r_i\epsilon$ with $r_i \in [0, 1]$; (iii) for three different random sets of perturbations of the mechanical constants up to 1% of κ and ρ, (iv) for random sets of perturbations of the mechanical constants up to 1%, 2%, 5%, 10%, of κ and ρ. [Colour online]

the physical system rather than the simulation itself. We return to modeling each switch as $\mu(u) = \tanh(u/\epsilon)$, but vary the damping and spring stiffness constants (the components of M^ρ and M^κ), by adding random values on the order of 1% in (iii), for three different sets of random perturbations, and of orders 1%, 2%, 5%, 10%, in (iv). For small perturbations in (iii) the results are similar to before, namely that short-time behaviour is sensitive and unpredictable, but long-term behaviour is robust and large-scale structures are preserved. In fig. 4(iv) we demonstrate that sufficiently large changes in the constants do yield a different system, with perturbations on the order of a few percent or more giving a very distinct system with large (period ~ 20) oscillations at 5%, and total collapse such that all oscillators are slipping at 10%.

To conclude let us explore this long-term dynamics a little further. Again we fix all $r_i = 1$ with $\epsilon = 0.03$ for the simulation. Now we shall perform simulations on a system of 200 oscillators, with different random matrices of damping and spring constants M^ρ and M^κ (these are randomly chosen, and not small perturbations of each other as in fig. 4(iii)). We simulate these for a significantly longer period of time, and observe in most cases (as in the three shown) that the complex stick-slip cascades persist over time in a self-sustaining manner, though occasional instances of constants can be found that collapse to $n_{stick} = 0$. The plots of sticking codimension for three sample cases are shown in fig. 5(i).

With so many oscillators and such long timescales, further insight is provided by considering the statistics of these self-sustaining stick slip events. We plot in fig. 5(ii) the size of cascades

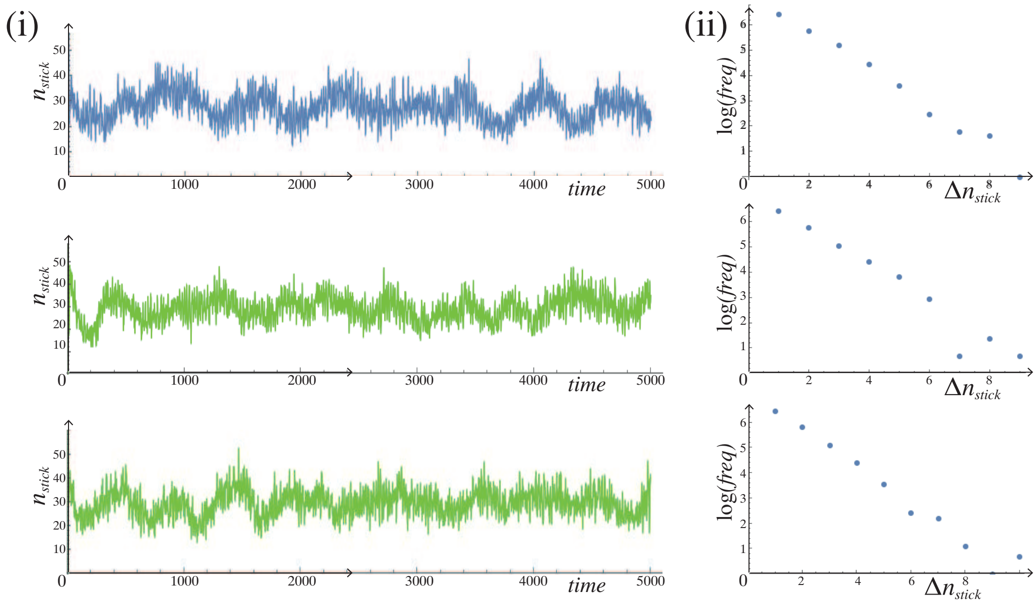

Figure 5: Plots of the sticking codimension n_{stick} for systems of 200 oscillators for three different sets of spring and damping coefficients in (i), and the frequency of stick-slip cascades of size Δn_{stick} in (ii) shows a logarithmic pattern for each system in (i).

Δn_{stick} (defined as the number of successive time increments in which the sticking codimension decreases), against the frequency of events of each size, showing that the frequency f of cascades of size Δn_{stick} fits a relation $f \propto e^{-\gamma \Delta n_{stick}}$ with $\gamma \approx 0.74$.

5 Hidden dynamics

What precisely goes on inside the sticking mode requires analysis beyond the standard Filippov/Utkin theory of sliding modes [5, 15], and must be determined by revealing the *hidden dynamics* [9], obtained by singular perturbation technique of performing a 'blow up' of the discontinuities. This also justifies the use of the sigmoid function in the simulations below, the stiffness parameters $r_i\epsilon$ playing the role of time scalings that give fast contraction (the fast element of the hidden dynamics) onto any sticking modes, see e.g. [11, 6].

The blow up analysis is performed as follows, and we express it in a general form. We relabel the different component vector fields in (1) by writing $\mathbf{f}^{\lambda_1 \ldots \lambda_m}(\mathbf{x}) = \mathbf{f}\left(\mathbf{x}; (\lambda_1, \ldots, \lambda_m)\right)$, so that the piecewise-smooth system becomes

$$\dot{\mathbf{x}} = \left\{ \mathbf{f}^{\lambda_1 \ldots \lambda_m}(\mathbf{x}) \;\; \text{where } \lambda_i = \operatorname{sign}\left(h_i\left(\mathbf{x}\right)\right) \right\} \; . \tag{11}$$

and then combine these in a *convex canopy* of values [8]

$$\mathbf{f} = \sum_{i_1, i_2, \ldots i_m = \pm} \lambda_1^{(i_1)} \lambda_2^{(i_2)} \ldots \lambda_m^{(i_m)} \mathbf{f}^{i_1 i_2 \ldots i_m} \; , \qquad \lambda_j^{(\pm)} \equiv (1 \pm \lambda_j)/2 \; . \tag{12}$$

If all oscillators are slipping then all h_j are nonzero, and all λ_j takes values ± 1. If some oscillators $j = 1, \ldots, r$ are sticking, we take coordinates $x_j = h_j$ for $j = 1, \ldots, r$, write $\mathbf{f}$ as $(f_1, \ldots, f_n)$, and form the blow up system by writing

$$\left\{ \begin{array}{rcl} (\lambda_1', \ldots, \lambda_r') & = & (f_1(\mathbf{x}; \boldsymbol{\lambda}) \quad, \ldots, f_r(\mathbf{x}; \boldsymbol{\lambda})) \; , \\ (\dot{x}_{r+1}, \ldots, \dot{x}_n) & = & (f_{r+1}(\mathbf{x}; \boldsymbol{\lambda}), \ldots, f_n(\mathbf{x}; \boldsymbol{\lambda})) \; . \end{array} \right. \tag{13}$$

The prime on the lefthand side denotes a fast dummy timescale, instantaneous on the timescale of the original system. This facilitates either the transition from one slipping direction to another, or, if equilibria of the fast subsystem exist, the λ_j's collapse in fast time to values λ_j^s given by

$$\left\{\begin{array}{rcl} (0,...,0) &=& (f_1(\mathbf{x};\boldsymbol{\lambda}^s) \quad ,...,f_r(\mathbf{x};\boldsymbol{\lambda}^s))\ , \\ (\dot{x}_{r+1},...,\dot{x}_n) &=& (f_{r+1}(\mathbf{x};\boldsymbol{\lambda}^s),...,f_n(\mathbf{x};\boldsymbol{\lambda}^s))\ , \end{array}\right. \tag{14}$$

where $\boldsymbol{\lambda}^s = (\lambda_1^s,...,\lambda_r^s,\lambda_{r+1},...,\lambda_n)$, and these correspond precisely to the usual notion of sticking modes deriving from [5].

In fig. 6 we illustrate the blow up system (13) applied to the two oscillator system in section 3. Panel (a) shows the typical situation inside the sticking parallelogram, when (z_1, z_2) take values such that a unique sticking mode exists, observed in the blow up as an equilibrium in the blow up dynamics on (μ_1, μ_2). As (z_1, z_2) vary outside the parallelogram shown in fig. 2, this equilibrium leaves the box $(\mu_1, \mu_2) \in [-1, +1] \times [-1, +1]$ depicted.

As an example of more novel hidden dynamics, in (6) let us replace the functions μ_1 and μ_2 by $\mu_1 - \mu_2(1-\mu_1^2)$ and $\mu_2 - \mu_1(1-\mu_2^2)$ respectively. During slip, when $\mu_{1,2} = \pm 1$, the additional nonlinear μ_i terms vanish, changing nothing in the original Coulomb friction problem. But in the sticking mode when $\mu_{1,2} \in [-1, +1]$ these new terms are non-vanishing, and signify a more complicated static friction interaction. These particular terms are chosen arbitrarily to demonstrate interesting dynamics, in practice their precise form would depend on refinements to the contact laws that we lack space to investigate here. The scenario shown in (b) illustrates an example where three solutions of (14) exist, giving three sticking modes, though only two are stable. Which of these the system attains depends on the entry trajectory, and can only be determined by full consideration of the system (13), not from the existence of sticking modes alone. Exit points will occur where either of the two stable equilibria leave the depicted region.

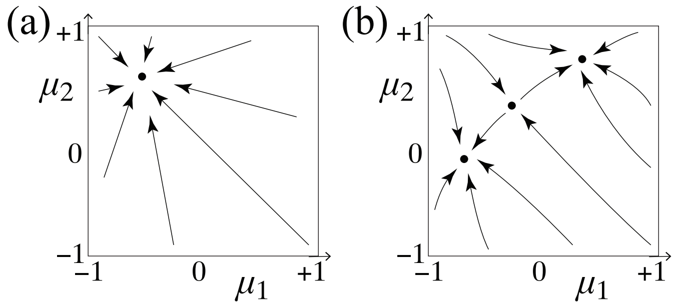

Figure 6: Two examples of hidden dynamics for the two oscillator system, showing: (a) attraction to a unique sticking solution, and (b) attraction to either of two possible sticking solutions (solutions of (14)).

6 Closing remarks

We have demonstrated a few key elements of recent piecewise-smooth dynamical systems theory that provide new insights into basic mechanisms lying behind complex dynamics in dry friction oscillators. This includes some surprises, particularly pertaining to self-sustaining stick-slip cascades, showing a regular logarithmic relation in the frequency of cascades of given size, and showing short-term unpredictability but long-term robustness to perturbation. Ongoing work will investigate further how the local geometry of exit points, shown here to underly global dynamical patterns, lead to the statistics and criticality thresholds observed, and begin applying these ideas to more specific large dimensional oscillator systems.

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