Hidden degeneracies in piecewise smooth dynamical systems

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(Dated: November 4, 2015)

When a flow suffers a discontinuity in its vector field at some switching surface, the flow can cross through or slide along the surface. Sliding can be understood as the flow along an invariant manifold inside a *switching layer*. It turns out that the usual method for finding sliding modes – the Filippov convex combination or Utkin equivalent control – results in a degeneracy in the switching layer whenever the flow is tangent to the switching surface from both sides. We derive the general result and analyse the simplest case here, where the flow curves parabolically on either side of the switching surface (the so-called fold-fold or two-fold singularities). The result is a set of zeros of the fast switching flow inside the layer, which is structurally unstable to perturbation by terms nonlinear in the switching parameter, terms such as $(\text{sign } x)^2$ [where the superscript *does* mean "squared"]. We provide structurally stable forms, and show that in this form the layer system is equivalent to a generic singularity of a two timescale system. Finally we show that the same degeneracy arises when a discontinuity is smoothed using the standard regularization methods.

I. INTRODUCTION

Discontinuous dynamical systems have long been of interest in impact mechanics and electronic switching, and they are increasingly being used to model changes in behaviour enacted at definite thresholds in a wide variety of engineering and biological applications (see e.g. [1, 6, 21, 23, 31]). Filippov [10] brought the subject to life by moving the emphasis from trajectories that cross switching surfaces, to developing an extensive dynamical theory around *sliding* trajectories, which evolve along switching surfaces. These have inspired a growing field of piecewise smooth dynamical theory (e.g. [6, 19, 28, 31] and references therein) that have proved of great utility for electronic control [7, 30].

Part of the importance of sliding orbits (or sliding modes) is their stability, both in providing robust control [7, 30], and in representing the average response of discontinuous switching in the presence of noise [25]. The boundaries of sliding regions are not constant, however, but change, either as we vary parameters or, in sufficiently high dimensions, as we move through phase space, see e.g. [10, 19, 28]. In this paper we will show that the boundaries of sliding exhibit a form of

degeneracy, and become structurally unstable with respect to perturbations we shall describe, at relatively simple singularities created when the flow is tangent to the switching surface from both sides.

In studying a discontinuous system

$$\dot{\mathbf{x}} = \begin{cases} \mathbf{f}^+(\mathbf{x}) & \text{if } h(\mathbf{x}) > 0 , \\ \mathbf{f}^-(\mathbf{x}) & \text{if } h(\mathbf{x}) < 0 , \end{cases} \tag{1}$$

for smooth vector fields $\mathbf{f}^\pm(\mathbf{x})$, we can seek a solvable deterministic system by continuing (1) across the switching surface $h = 0$ as a convex combination

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}; \lambda) = \frac{1+\lambda}{2}\mathbf{f}^+(\mathbf{x}) + \frac{1-\lambda}{2}\mathbf{f}^-(\mathbf{x}) , \tag{2}$$

where $\mathbf{f}(\mathbf{x}; \pm 1) \equiv \mathbf{f}^\pm(\mathbf{x})$ and

$$\begin{cases} \lambda = \operatorname{sign} h & \text{if } h \neq 0 , \\ \lambda \in (-1, +1) & \text{if } h = 0 . \end{cases} \tag{3}$$

The rigour and depth of the theory that takes (2) as its starting point, see for example [6, 10, 18, 28], warrant a deeper consideration of the extent to which the prescription (2) can be considered a robust dynamical model of the piecewise-defined system (1).

The present paper aims to highlight certain conditions in which such systems are structurally *unstable* under close inspection at the discontinuity, yet they appear to be structurally *stable* from the expression (2) alone. The conditions are those defining singularities that often involve forward and/or backward time non-uniqueness of solutions, making issues of stability particularly important to resolving their behaviour.

The instability arises in resolving the switching surface $h = 0$, either by blowing up the discontinuity spatially into a non-vanishing strip, or smoothing it out by means of some sigmoidal interpolation. In either case, an invariant manifold associated with sliding motion along the surface may become degenerate. This occurs around tangencies between the vector fields in (1) and the surface $h = 0$, creating a structurally unstable set of zeros in the transition across the discontinuity. This is introduced in section II.

In section III we show that the degeneracy is broken by admitting in (2) terms nonlinear in λ. The possibility of nonlinear λ dependence can be found discussed already in [10, 24, 30] (including an experimental model in [24]), and here we find them taking on a necessary role in the local qualitative classification of nonsmooth systems.

The situations resulting in degeneracy include the fold-fold singularities classified in [6, 10, 19], and the two-fold or fold-cusp singularities classified in [2, 10, 28], for all of which normal form expressions have been established that appear from the expression (2) to be structurally stable. Thus as we extend the depth of our analysis into piecewise smooth systems, so we extend our understanding of their structural stability.

We reconsider the simplest of these, the fold-fold bifurcation in planar systems, and the closely related two-fold singularity in higher dimensional systems, in sections IV-V. A detailed study of the different classes of behaviour that result is a lengthy exercise beyond our scope here, we detail only how the degeneracy arises, how it is resolved, and we outline the starting point for further study. For the generic form of the two-fold, for example, the structurally stable system is shown to be equivalent to a corresponding generic singularity in smooth two timescale systems; while the superficial resemblance of these two singularities has been obvious, the exact relationship between them has been unclear until now.

In section VI we show that equivalent results hold if the discontinuity is regularized by smoothing. Some final remarks are given in section VII.

II. THE DEGENERACY

The degeneracy that concerns us in (2) arises as follows. Take coordinates $\mathbf{x} = (x, y, z, ...)$ in which $h = x$, and for brevity write $X = (y, z, ...)$ so that $\mathbf{x} = (x, X)$, and similarly we write $\mathbf{f} = (f, F)$ where $f \in \mathbb{R}$, $F \in \mathbb{R}^{n-1}$.

Standard piecewise smooth theory for a Filippov system like (2) asks whether there exist solutions of (2) that slide along the switching surface $h = 0$, thus satisfying $\dot{h} = 0$. Using $\dot{h} = \mathbf{f} \cdot \nabla h = f$, sliding trajectories satisfy the differential-algebraic system

$$
\begin{aligned}
0 &= f(0, X; \lambda) , \\
\dot{X} &= F(0, X; \lambda) ,
\end{aligned}
\tag{4}
$$

on $x = 0$ for some $\lambda \in (-1, +1)$. If no such solutions exist then all vectors $\{\mathbf{f}(0, X; \lambda) : \lambda \in (-1, +1)\}$ direct the flow transversally across the switching surface. Solutions to (4) exist when $f^+(0, X)$ and $f^-(0, X)$ have opposite signs, so that both vector fields $\mathbf{f}^\pm$ push the flow onto the surface or both pull it away, then the second line of (4) specifies so-called *sliding dynamics* along $x = 0$, with $\lambda = \frac{f^-(0,X) + f^+(0,X)}{f^-(0,X) - f^+(0,X)}$ lying in the interval $(-1, +1)$.

The sliding problem (4) is a subsystem of a *switching layer* system

$$
\begin{aligned}
\varepsilon\dot{\lambda} &= f(0, X; \lambda) , \\
\dot{X} &= F(0, X; \lambda) ,
\end{aligned}
\tag{5}
$$

on $x = 0$, where $\varepsilon \geq 0$ is an infinitesimal singular perturbation parameter. The first line blows up the jump in λ between ± 1 into a fast dynamical system on the interval $(-1, +1)$. (In section VI we describe the relation of this layer system, proposed in [11], to a regularization derived by replacing λ in (2) with a smooth sigmoid function according to the Sotomayor-Teixeira method [26]; in particular we show that the results which follow also apply to such systems).

The system (5) now provides a two-timescale dynamical system specifying how the flow evolves through the discontinuity. With respect to a fast timescale $\tau = t/\varepsilon$, (5) becomes

$$
\begin{aligned}
\lambda' &= f(0, X; \lambda) , \\
X' &= \varepsilon F(0, X; \lambda) ,
\end{aligned}
\tag{6}
$$

with the prime denoting differentiation with respect to the fast time τ, and this becomes a one dimensional fast subsystem in the limit $\varepsilon \to 0$,

$$
\begin{aligned}
\lambda' &= f(0, X; \lambda) , \\
X' &= 0 .
\end{aligned}
\tag{7}
$$

This describes how λ either transitions directly between ± 1, or else encounters zeros of $f(0, X; \lambda)$, in the limit as $\varepsilon \to 0$.

The equilibria of (6) occupy a hypersurface which we call the *sliding manifold*

$$
\mathcal{M} = \left\{ (\lambda, X) \in (-1, +1) \times \mathbb{R}^{n-1} : 0 = f(0, X; \lambda) \right\} .
\tag{8}
$$

These equilibria are normally hyperbolic except at a set of points

$$
\mathcal{L} = \left\{ (\lambda, X) \in \mathcal{M} : f_{,\lambda}(0, X; \lambda) = 0 \right\} ,
\tag{9}
$$

where $f_{,\lambda}$ denotes $\partial f / \partial \lambda$. The sliding manifold $\mathcal{M}$ is just the surface defined by the algebraic condition in (4), making $\mathcal{M} \backslash \mathcal{L}$ an invariant manifold of (4). In geometric singular perturbation theory, $\mathcal{M}$ is known as the critical manifold of the system (5), and the existence of invariant manifolds near $\mathcal{M} \backslash \mathcal{L}$ in the $\varepsilon \neq 0$ system then follows by the theory of Fenichel, see [8, 16].

Our concern here, however, is on the critical limit $\varepsilon = 0$. In particular we are concerned with the fact that if a set $\mathcal{L}$ occurs in the system as defined above, then it turns out that it is always degenerate.

Proposition 1. *In the switching layer system (5) using the convex combination (2), the non-hyperbolic set $\mathcal{L}$ of the sliding manifold $\mathcal{M}$ is degenerate.*

Proof. Let us expand the first line of (5) as a series in λ about some $\lambda_0 \in (-1, +1)$,

$$f(0, X; \lambda) = f(0, X; \lambda_0) + (\lambda - \lambda_0) f_{,\lambda}(0, X; \lambda_0) + \mathsf{O}\left((\lambda - \lambda_0)^2\right) . \tag{10}$$

On the set $\mathcal{L}$ the two lowest order terms in this expression vanish, leaving

$$f(0, X; \lambda) = \mathsf{O}\left((\lambda - \lambda_0)^2\right) . \tag{11}$$

This vanishes identically for $\mathbf{f}$ as defined in (2) because f depends only linearly on λ, so the first line of (4) or (5) is trivial on $\mathcal{L}$, and the value of λ is undetermined. Geometrically this means that $\mathcal{L}$ lies along the λ coordinate direction, and the fast direction of the corresponding perturbed system (5), and represents a line of zeros of the fast subsystem (7). $\qquad\square$

The condition for $\mathcal{L}$ to lie transversal to the fast direction is simply that the next derivative of f is nonzero, i.e. $\partial^2/\partial\lambda^2 f \neq 0$ in (9). Perturbing the expression (2) with a term nonlinear in λ will therefore yield a topologically distinct system. In section III we discuss the forms of such perturbations that are consistent with (1).

The significance of singularities that create a non-hyperbolic set $\mathcal{L}$ is seen by considering what happens when $\mathcal{L} \subset \mathcal{M}$ meets the boundaries of the interval $\lambda \in (-1, +1)$ at a given value of X (as it is bound to do if, as above, $\mathcal{L}$ lies along the λ direction). The condition for $\mathcal{M}$ at the boundaries is $0 = f(0, X; \pm 1) \equiv f^{\pm}(0, X)$, i.e. that the components $f^{\pm}(0, X)$ normal to the switching surface vanish, meaning the vector fields $\mathbf{f}^{\pm}(\mathbf{x})$ are both tangent to the switching surface at a given point $\mathbf{x} = (0, X)$.

The outcome of the argument above is that whenever tangencies between $\mathbf{f}^{\pm}$ and $h = 0$ coincide in (1), a set $\mathcal{L}$ exists for (2), but in a structurally unstable form. Certain classes of perturbations should break the degeneracy so that the non-hyperbolic set on $\mathcal{M}$ lies in a general position with respect to the flow. In the rest of this paper we shall explore what kinds of perturbations these are, when non-hyperbolic sets occur, and particularly where they can be found in familiar singularities, and we shall see how to augment (2) so that under closer inspection it remains structurally stable.

III.   EXPANSION IN TERMS OF λ

A generalization of (2) was proposed in [11] which in the present context can be derived as follows. If we expand $\mathbf{f}$ about $\lambda = +1$ we have

$$\mathbf{f}(\mathbf{x};\lambda) \ = \ \mathbf{f}(\mathbf{x};+1) + (\lambda - 1)\mathbf{f}_{,\lambda}(\mathbf{x};+1) + \tfrac{1}{2}(\lambda - 1)^2\mathbf{f}_{,\lambda\lambda}(\mathbf{x};+1) + \mathsf{O}\left((\lambda - 1)^3\right) := P \qquad (12)$$

while expanding about $\lambda = -1$ gives

$$\mathbf{f}(\mathbf{x};\lambda) \ = \ \mathbf{f}(\mathbf{x};-1) + (\lambda + 1)\mathbf{f}_{,\lambda}(\mathbf{x};-1) + \tfrac{1}{2}(\lambda + 1)^2\mathbf{f}_{,\lambda\lambda}(\mathbf{x};-1) + \mathsf{O}\left((\lambda + 1)^3\right) := Q \ . \qquad (13)$$

A function satisfying both of these expansions can be found by taking the convex combination of the two approximations, $\frac{1+\lambda}{2}P + \frac{1-\lambda}{2}Q$, which gives

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x};\lambda) = \frac{1+\lambda}{2}\mathbf{f}^{+}(\mathbf{x}) + \frac{1-\lambda}{2}\mathbf{f}^{-}(\mathbf{x}) + (\lambda^2 - 1)\mathbf{g}(\mathbf{x};\lambda) \ , \qquad (14)$$

where

$$\mathbf{g}(\mathbf{x};\lambda) = \frac{\mathbf{f}_{,\lambda}(\mathbf{x};+1) - \mathbf{f}_{,\lambda}(\mathbf{x};-1)}{2} - \frac{1+\lambda}{4}\mathbf{f}_{,\lambda\lambda}(\mathbf{x};-1) - \frac{1-\lambda}{4}\mathbf{f}_{,\lambda\lambda}(\mathbf{x};+1) + \ldots \ .$$

The expression (11) will now typically be nontrivial, with a non-vanishing second order term

$$f(0,X;\lambda) = \tfrac{1}{2}(\lambda - \lambda_0)^2 f_{,\lambda\lambda}(0,X;\lambda_0) + \mathsf{O}\left((\lambda - \lambda_0)^3\right) \qquad (15)$$

where

$$f_{,\lambda\lambda}(0,X;\lambda_0) = 2g(0,X;\lambda_0) + 4\lambda_0 g_{,\lambda}(0,X;\lambda_0) + (\lambda_0^2 - 1)g_{,\lambda\lambda}(0,X;\lambda_0) \ .$$

In a system where $g = g_0 + \mathsf{O}\left(\lambda - \lambda_0\right)$, with $f_{,\lambda\lambda}(0,X;\lambda_0) = 2g_0$ for a nonzero constant g_0, if a non-hyperbolic set $\mathcal{L} \subset \mathcal{M}$ exists then it will typically be non-degenerate.

We can then think of (14) as a series expansion simultaneously about the points $\lambda = \pm 1$. The expression (2) therefore belongs to a larger class of functions $\mathbf{f}(\mathbf{x};\lambda)$ given by (14) that are consistent with (1), where $\mathbf{g}$ is some vector field that is smooth in $\mathbf{x}$ and λ. The factor $\lambda^2 - 1$ or something similar must clearly always appear for consistency with (1), because any term added to (2) must vanish for $\lambda = \pm 1$. In fact, if we assume $\mathbf{f}$ can be written as a polynomial in λ then it can always be cast in the form (15) [12, 15]. The nonlinear term $(1 - \lambda^2)\mathbf{g}$ is sometimes called a 'hidden' term because it vanishes everywhere except at the discontinuity.

In the following sections we will analyse the set $\mathcal{L}$ in the most basic situation where it exhibits a degeneracy, in two or three dimensions, and we will show the effect of introducing a non-vanishing nonlinear term, with a constant $\mathbf{g} = (g_0, 0, 0, \ldots)$.

IV. THE PLANAR FOLD-FOLD SINGULARITY

Definition 1. *A fold singularity in a piecewise smooth system (1) is a point $\hat{\mathbf{x}} \in \mathbb{R}^n$ where the vector field $\mathbf{f}^+$ or $\mathbf{f}^-$ has quadratic contact with the switching manifold $h = 0$, that is where $\mathbf{f}^+(\hat{\mathbf{x}}) \cdot \nabla h(\hat{\mathbf{x}}) = h(\hat{\mathbf{x}}) = 0$ or $\mathbf{f}^-(\hat{\mathbf{x}}) \cdot \nabla h(\hat{\mathbf{x}}) = h(\hat{\mathbf{x}}) = 0$, and the corresponding second derivatives $(\mathbf{f}^\pm \cdot \nabla)^2 h$ are nonzero.*

A fold is described as *visible* if it curves away from the switching surface $(\mathbf{f}^+ \cdot \nabla h = h = 0 < (\mathbf{f}^+ \cdot \nabla)^2 h$ or $\mathbf{f}^- \cdot \nabla h = h = 0 > (\mathbf{f}^- \cdot \nabla)^2 h)$, and *invisible* if it curves away from the switching surface $(\mathbf{f}^+ \cdot \nabla h = h = 0 > (\mathbf{f}^+ \cdot \nabla)^2 h$ or $\mathbf{f}^- \cdot \nabla h = h = 0 < (\mathbf{f}^- \cdot \nabla)^2 h)$.

Definition 2. *A fold-fold singularity in a system (1) is a point $\hat{\mathbf{x}} \in \mathbb{R}^2$ where*

$$h(\hat{\mathbf{x}}) = \mathbf{f}^+(\hat{\mathbf{x}}) \cdot \nabla h(\hat{\mathbf{x}}) = \mathbf{f}^-(\hat{\mathbf{x}}) \cdot \nabla(\hat{\mathbf{x}}) = 0, \tag{16}$$

and $(\mathbf{f}^\pm(\hat{\mathbf{x}}) \cdot \nabla)^2 h(\hat{\mathbf{x}}) \neq 0$.

The different forms the fold-fold can take were classified in [10], and normal forms for each class were proposed in [19], which we can represent in a unified form

$$(\dot{x}, \dot{y}) = \begin{cases} (\beta + y, b_1 + b_2 y) & \text{if } x > 0, \\ (cy, d_1 + d_2 y) & \text{if } x < 0, \end{cases} \tag{17}$$

where β is a bifurcation parameter and b_1, b_2, c, d_1, d_2, are constants. This amounts to the leading order term in an expansion about a fold at $(x, y) = (0, -\beta)$ for $x > 0$ and a fold at $(x, y) = (0, 0)$ for $x < 0$.

The convex combination (2) for (17) is

$$(\dot{x}, \dot{y}) = \frac{1 + \lambda}{2} (\beta + y, b_1 + b_2 y) + \frac{1 - \lambda}{2} (cy, d_1 + d_2 y), \tag{18}$$

giving a switching layer system (5) on $x = 0$ for infinitesimal $\varepsilon \geq 0$,

$$(\varepsilon \dot{\lambda}, \dot{y}) = \frac{1 + \lambda}{2} (\beta + y, b_1 + b_2 y) + \frac{1 - \lambda}{2} (cy, d_1 + d_2 y). \tag{19}$$

The sliding manifold (8), on which $\dot{\lambda} = 0$, is

$$\mathcal{M} = \left\{ (\lambda, y) \in (-1, +1) \times \mathbb{R} : \lambda = \frac{(c + 1)y + \beta}{(c - 1)y - \beta} \right\}. \tag{20}$$

Hyperbolicity of $\mathcal{M}$ requires $\partial \dot{\lambda} / \partial \lambda \neq 0$, and breaks down on the set

$$\mathcal{L} = \{ (\lambda, y) \in \mathcal{M} : (1 + \lambda)(\beta + y) + (1 - \lambda)cy = \beta + (1 - c)y = 0 \}, \tag{21}$$

which only has solutions for $y = 0$ and $\beta = 0$, for any $\lambda \in (-1, +1)$. The fast system on $\mathcal{L}$ in the limit $\varepsilon = 0$ is then

$$(\lambda', y') = \frac{1+\lambda}{2}(\beta + y, 0) + \frac{1-\lambda}{2}(cy, 0) \ , \tag{22}$$

which is a y-parameterized family of equilibria of the fast λ subsystem. However, at the parameter values $\beta = y = 0$ the entire righthand side of (22) vanishes identically for any λ. Therefore $\mathcal{L}$ is a line of singular points. Its appearance coincides with the bifurcation that takes place at $\beta = 0$.

Because the leading order expression (17) leads to a degeneracy whereby the fast system vanishes at $\beta = 0$ on $\mathcal{L}$, we introduce the next highest order term from the x component of the extended expression (14),

$$(\dot{x}, \dot{y}) = \frac{1+\lambda}{2}(\beta + y, b_1 + b_2 y) + \frac{1-\lambda}{2}(cy, d_1 + d_2 y) + (1 - \lambda^2)(\alpha, 0) \ , \tag{23}$$

where α is some constant. The switching layer system (5) becomes

$$(\varepsilon\dot{\lambda}, \dot{y}) = \frac{1+\lambda}{2}(\beta + y, b_1 + b_2 y) + \frac{1-\lambda}{2}(cy, d_1 + d_2 y) + (1 - \lambda^2)(\alpha, 0) \ , \tag{24}$$

with a sliding manifold (8)

$$\mathcal{M} = \left\{ (\lambda, y) \in (-1, +1) \times \mathbb{R} \ : \ 0 = (\beta + y) + cy + \lambda(\beta + y - cy) + (1 - \lambda^2)\alpha \right\} \ . \tag{25}$$

The non-hyperbolic set now consists of well-defined isolated points

$$\mathcal{L} = \left\{ (\lambda, u \in \mathcal{M} \ : \ \lambda = \frac{c+1}{c-1} \pm \sqrt{\frac{c^2+1}{(c-1)^2} - \frac{2c\alpha\beta}{c-1}}, \ y = \frac{2\lambda\alpha - \beta}{1 - c} \right\} \tag{26}$$

of which one, two, or zero may exist such that $-1 < \lambda < +1$. Thus $\mathcal{L}$ is typically non-degenerate, except for a well-defined bifurcation that will occur when the square root in (26) vanishes.

As an example consider

$$(\dot{x}, \dot{y}) = \begin{cases} (\beta + y, 2) & \text{if } x > 0 \ , \\ (-y, -1) & \text{if } x < 0 \ , \end{cases} \tag{27}$$

whose phase portrait in (x, y) space is illustrated in figure 1. The switching surface is attracting for $y < \min(0, -\beta)$ and repelling for $y > \max(0, -\beta)$, the folds lying at $y = 0$ and $y = -\beta$. The basic phase portrait is depicted in figure 1, showing the movement of the folds through a bifurcation at $\beta = 0$. A node-like equilibrium exists in the sliding dynamics on the surface, and its precise character must be determined by closer inspection of the switching layer. This system is studied more closely in [13], we shall analyse only the appearance of $\mathcal{L}$ for this system.
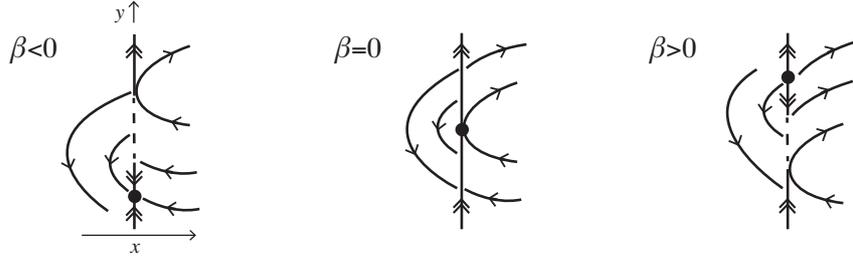
FIG. 1: A bifurcation in which the ('curving away from $x = 0$') and invisible ('curving towards $x = 0$') folds in the flow exchange ordering, is accompanied by a (pseudo)-node changing from attracting to repelling.
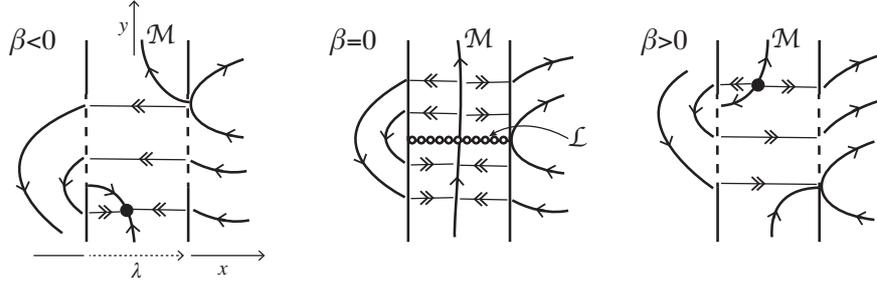
FIG. 2: The bifurcation in figure 1, with $x = 0$ blown up into the layer $\lambda \in (-1, +1)$.

The switching layer is depicted in figure 2 for $\alpha = 0$, showing the bifurcation that occurs at $\beta = 0$, corresponding to a bifurcation in the sliding manifold $\mathcal{M} = \left\{ (\lambda, y) \in \mathcal{M} \ : \ \lambda = \frac{-\beta}{2y+\beta} \right\}$, with the formation of a structurally unstable non-hyperbolic set $\mathcal{L}$.

For $\alpha \neq 0$ the non-hyperbolic set is instead a pair of points

$$\mathcal{L} = \left\{ (\lambda, y) \in \mathcal{M} \ : \ \lambda = \pm\sqrt{\tfrac{1}{2} - \alpha\beta}, \ y = \lambda\alpha - \tfrac{1}{2}\beta \right\}$$

that exist for $\beta < 1/2\alpha$, coalesce when $\beta = 1/2\alpha$ and then vanish for $\beta > 1/2\alpha$. The case illustrated in figure 3 is for $\alpha < 0$.
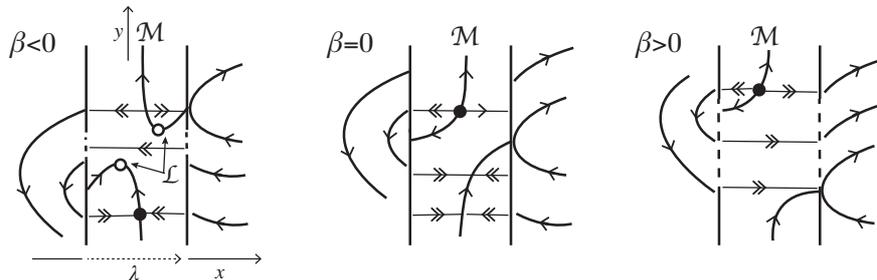
FIG. 3: The bifurcation in figure 1 for $\beta > 0$, with $x = 0$ blown up into the layer $\lambda \in (-1, +1)$. .

The example above is only one of several scenarios of fold-fold bifurcations in the plane represented by (17), each described in [10, 19]. Each of these involve the same degeneracy in the

switching layer, which in certain cases (including the example above) leads to different layer dynamics for $\alpha > 0$ and $\alpha < 0$, as can be found in [13].

In three dimensions or more, similar coincidences of folds occur as generic singularities (i.e. not as bifurcations), and are then known as two-fold singularities. They give rise to far more intricate dynamics in which the degeneracy of $\mathcal{L}$ takes on greater significance. The basic geometry, which has not been studied before, is analysed in the next section.

V.   THE TWO-FOLD SINGULARITY

Definition 3. *A two-fold singularity is a point* $\hat{\mathbf{x}} \in \mathbb{R}^{n \geq 3}$ *where*

$$h(\hat{\mathbf{x}}) = \mathbf{f}^+(\hat{\mathbf{x}}) \cdot \nabla h(\hat{\mathbf{x}}) = \mathbf{f}^-(\hat{\mathbf{x}}) \cdot \nabla h(\hat{\mathbf{x}}) = 0, \tag{28}$$

such that $(\mathbf{f}^+(\hat{\mathbf{x}}) \cdot \nabla)^2 h(\hat{\mathbf{x}}) \neq 0$ *and the vectors* $\nabla h(\hat{\mathbf{x}})$, $\nabla \left(\mathbf{f}^+(\hat{\mathbf{x}}) \cdot \nabla h(\hat{\mathbf{x}}) \right)$, $\nabla \left(\mathbf{f}^-(\hat{\mathbf{x}}) \cdot \nabla h(\hat{\mathbf{x}}) \right)$ *are linearly independent.*

The normal form of the two-fold singularity commonly found in the literature (e.g. [2, 10, 27]) is

$$(\dot{x}, \dot{y}, \dot{z}) = \begin{cases} (-y, a_1, b_1) & \text{if } x > 0 \,, \\ (+z, b_2, a_2) & \text{if } x < 0 \,, \end{cases} \tag{29}$$

in terms of constants $a_i = \pm 1$ and $b_i \in \mathbb{R}$. By results in [2, 10, 28], a system is locally approximated by (29) when it satisfies the conditions in Definition 3.

The two-fold is a generic singularity in piecewise smooth systems of three or more dimensions. Nevertheless, establishing its stability has been a topic of considerable intrigue since [10, 27]. It is known to exist in a number of forms with bifurcations between them [2]. Within these forms, its phase portrait obtained by substituting (29) into (2) appears to be structurally stable. The analysis of section II shows, however, that this does not hold inside the switching layer, where the phase portrait will not be stable to perturbations of the form (14) .

The conventional qualitative picture, taking (29) with (2), is as shown in figure 4. The local flow 'folds' towards or away from the switching surface $x = 0$, determined by a_1 and a_2, along the line $y = x = 0$ on one side of the surface, and along the line $z = x = 0$ on the other. The point where these fold lines cross is the '*two*-fold'. As a result, the surface $x = 0$ is attractive in $y, z > 0$ and repulsive in $y, z < 0$, while trajectories cross the surface transversely in $yz < 0$. The three main flavours of two-fold are therefore: the visible two-fold for $a_1 = a_2 = -1$, the invisible two-fold
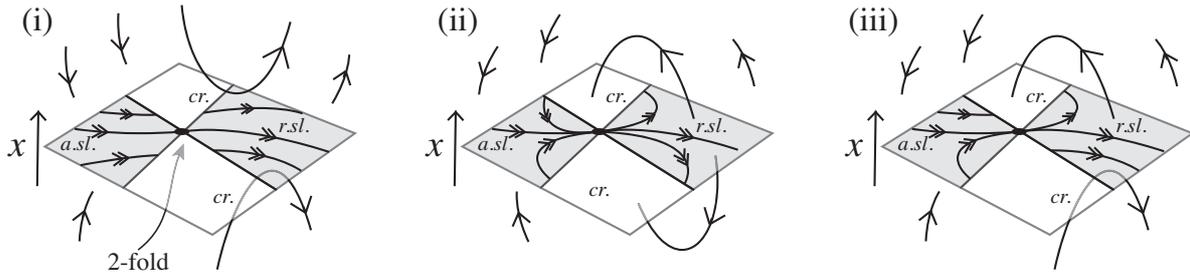
FIG. 4: Three kinds of two-fold formed by tangencies between the flow and the switching surface, of visible ('curving away') and invisible ('curving towards') type on either side. In the unshaded regions the flow crosses ($cr.$) through a discontinuity at $x = 0$, in the shaded regions the flow can only *slide* along the discontinuity on $x = 0$, the sliding region being attracting ($a.sl.$) or repelling ($r.sl.$) with respect to the flows outside the surface.

for $a_1 = a_2 = 1$, and the mixed two-fold for $a_1 a_2 = -1$; an example of each is shown in figure 4 (i,ii,iii) respectively. As in the previous section, the terms visible or invisible indicate that the flow is curving away from or towards the discontinuity surface, respectively.

In the attractive and repulsive regions the flow slides along the surface $x = 0$, following the vector field that is found by substituting (29) into (2) or (14), and solving for λ such that $\dot{x} = 0$. The precise form of both the crossing and sliding dynamics also depends on the constants b_1 and b_2, and an accounting of the classes of dynamics that arise from these simple features has been studied in many papers; a summary and references can be found in [2].

We shall henceforth be concerned only with the dynamics that is missing from the picture above, namely the dynamics inside the switching surface at the singularity itself, connecting the attractive and repelling branches of sliding.

We first form a system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}; \lambda)$ for (29) (using either (2) or (14)). The switching layer system is given by (5), and the sliding subsystem given by (4) on the manifold $\mathcal{M}$ defined in (8). The sliding manifold $\mathcal{M}$ is normally hyperbolic except on the curve $\mathcal{L}$ given by (9).

A. The unperturbed system

The two-fold normal form (29) substituted into the convex combination (2) is

$$
\begin{aligned}
(\dot{x}, \dot{y}, \dot{z}) &= \frac{1+\lambda}{2}(-y, a_1, b_1) + \frac{1-\lambda}{2}(z, b_2, a_2) \\
&:= (f(x, y, z; \lambda),\ F_2(x, y, z; \lambda),\ F_3(x, y, z; \lambda)) ,
\end{aligned}
\tag{30}
$$

recalling $\mathbf{x} = (x, y, z)$ and $\mathbf{f} = (f, F) = (f, F_2, F_3)$. The switching layer system (5) on $x = 0$ is

$$
(\varepsilon\lambda, \dot{y}, \dot{z}) = (f(x, y, z; \lambda),\ F_2(x, y, z; \lambda),\ F_3(x, y, z; \lambda)) .
\tag{31}
$$

By (8), the sliding manifold $\mathcal{M}$ is

$$\mathcal{M} = \left\{ (\lambda, y, z) \in (-1, +1) \times \mathbb{R}^2 : \ \lambda = \frac{z - y}{z + y} \right\} . \tag{32}$$

The condition $|\lambda| < 1$ implies that $\mathcal{M}$ exists for $0 < yz$, corresponding to attracting or repelling regions of the switching surface.

By expressing $\mathcal{M}$ implicitly as the zero contour of the smooth function $(z + y)\lambda + (y - z)$, we see that it is smooth surface which twists over near $y = z = 0$. It consists of two normally hyperbolic branches, one attractive in $y, z > 0$ since $\partial f / \partial \lambda = -(z + y)/2 < 0$), and one repulsive in $y, z < 0$ since $\partial f / \partial \lambda = -(z + y)/2 > 0$. The two branches are connected at $y = z = 0$ along the non-hyperbolic set, found from (9) to be

$$\mathcal{L} = \{ (\lambda, y, z) \in \mathcal{M} : \ y = z = 0 \} . \tag{33}$$

This line segment $\mathcal{L}$, at which the attracting and repelling branches of $\mathcal{M}$ intersect, constitutes the blowing up of the two-fold singularity ($x = y = z = 0$ in (29)) into a curve at $y = z = 0$ with $|\lambda| < 1$. Figure 5 shows an example of the piecewise smooth system in (i), the switching layer showing $\mathcal{M}$ and $\mathcal{L}$ in (ii), which is then rotated in to show $\mathcal{L}$ more clearly in (iii).
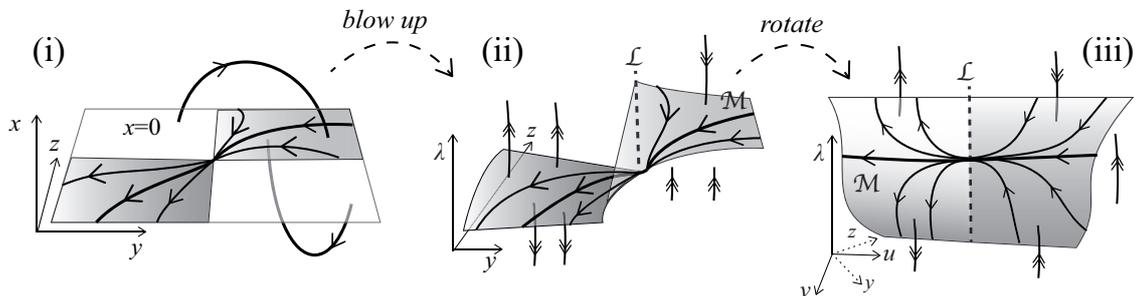
FIG. 5: The switching layer for the unperturbed system (30), for the example of an invisible two-fold. (i) The flow directions outside $x = 0$ create an attracting sliding region in $y, z > 0$ and repelling sliding region in $y, z < 0$. (ii) The switching layer on $x = 0$, where the sliding regions create a sliding manifold $\mathcal{M}$ (shaded), hyperbolic except along the vertical line $\mathcal{L}$, which aligns with the fast (double arrowed) λ dynamics. (iii) The dynamics in the manifold is best viewed along the v axis of rotated coordinates $u = y + z$, $v = y - z$.

Proposition 1 therefore holds, in the particular form:

Corollary 2. *In the switching layer system (5) of the normal form two-fold singularity (29), the non-hyperbolic set $\mathcal{L}$ of the sliding manifold $\mathcal{M}$ lies everywhere tangent to the coordinate axis of the fast variable.*

Proof. The non-hyperbolic set $\mathcal{L}$ forms a line with tangent vector $e_{\mathcal{L}} = (1,0,0)$ in the space of (λ, y, z), which means it lies everywhere parallel to the fast u-coordinate axis of the two timescale system (31). $\square$

As in the main proposition, this degeneracy is related to the fact that all derivatives of f with respect to the fast variable u vanish along $\mathcal{L}$, not only the first derivative $\partial f / \partial \lambda = -z - y = 0$ which defines $\mathcal{L}$ as the set $y = z = 0$, but also all higher derivatives $\partial^r f / \partial \lambda^r = 0$ for any $r > 1$. Thus this constitutes an infinite codimension degeneracy.

First observe that adding constant terms or functions of the coordinates (x, y, z) to (30) would only move the set $\mathcal{L}$ in the (y, z) plane, not remove its degeneracy, easily seen since the derivatives

$$\frac{\partial^r}{\partial \lambda^r} f(0, y, z; \lambda) = \frac{f(0, y, z; +1) - f(0, y, z; -1)}{2} \quad \text{for} \quad r > 0$$

would still vanish on $\mathcal{L}$, where $f(0, y, z; +1) - f(0, y, z; -1) = 0$. The only recourse to break the degeneracy, specifically to give $\frac{\partial^2 f}{\partial \lambda^2} \neq 0$, is therefore to add terms nonlinear in λ to (30). Anything we add to the function f in (30) must still give (29), so it must vanish outside the switching surface $x = 0$, i.e. be a perturbation in the form (14). We shall show that perturbing $\dot{x}$ with a term proportional to $\lambda^2 - 1$ is sufficient for structural stability. Perturbing $\dot{y}$ or $\dot{z}$ is neither necessary nor sufficient, therefore we leave them unaltered.

B.   The perturbed system

We now show that the system

$$(\dot{x}, \dot{y}, \dot{z}) = \frac{1 + \lambda}{2} (-y, a_1, b_1) + \frac{1 - \lambda}{2} (z, b_2, a_2) + (1 - \lambda^2)(\alpha, 0, 0) \tag{34}$$
$$:= (f(x, y, z; \lambda), F_2(x, y, z; \lambda), F_3(x, y, z; \lambda)) \,,$$

where α is a constant, does not suffer the structural stability of (30), and more interestingly we explore the consequences of removing the degeneracy. The switching layer system (5) becomes

$$(\varepsilon \dot{x}, \dot{y}, \dot{z}) = (f(x, y, z; \lambda), F_2(x, y, z; \lambda), F_3(x, y, z; \lambda)) \,. \tag{35}$$

Our main result is as follows:

Proposition 3. *The switching layer system (35) of the normal form two-fold singularity (29), using (14) with $\mathbf{g} = (\alpha, 0, 0)$, can be transformed into*

$$\varepsilon \dot{x} = y + x^2 + \mathsf{O}\left(\varepsilon x, \varepsilon z, xz\right)$$
$$\dot{y} = pz + qx + \mathsf{O}\left(z^2, xz\right)$$
$$\dot{z} = r + \mathsf{O}\left(z, x\right)$$

provided $\alpha \neq 0$ *for small* $\varepsilon > 0$, *where* p, q, r, *are real constants, and provided the conditions* $\frac{1}{2}(b_1 - b_2) \leq 1 = a_1 = -a_2$ *or* $\frac{1}{2}(b_1 - b_2) \geq -1 = a_1 = -a_2$ *do not hold.*

The significance of this result is that the system in the proposition is the canonical form for a generic singularity in a smooth two timescale system, arising at the transversal intersection of attracting and repelling invariant manifolds responsible for so-called canards [32]. It turns out that the cases excluded by the conditions $\pm\frac{1}{2}(b_1 - b_2) \leq 1 = \pm a_1 = \mp a_2$ are those in which there are no orbits of the sliding flow passing through the singularity, so the repelling and attracting branches of sliding are not directly connected by the flow.

The proof of the proposition is a fairly lengthy coordinate transformation that untwists $\mathcal{M}$ and folds it into a parabolic surface. To emphasise the interesting geometry we prove the proposition by way of three lemmas, establishing first the non-degeneracy of $\mathcal{L}$, second locating a new singularity that distinguishes a special point along $\mathcal{L}$, and finally morphing $\mathcal{M}$ into the $\dot{x}$ nullcline of the system in Proposition 3. At the end of the section we review the basic analysis used to classify the resulting singularity, and review where these results fit into previous studies of the two-fold.

The sliding manifold, found by applying (8) to (35), is now the set

$$\mathcal{M} = \left\{ (\lambda, y, z) \in (-1, +1) \times \mathbb{R}^2 \; : \; \frac{1-\lambda}{2}z - \frac{1+\lambda}{2}y + \alpha(1 - \lambda^2) = 0 \right\} , \qquad (36)$$

which is normally hyperbolic except on the set given by applying (9) to (35),

$$\mathcal{L} = \left\{ (\lambda, y, z) \subset \mathcal{M} : \; \lambda = 2\frac{2\alpha + z - y}{z + y} = -\frac{z + y}{4\alpha} \right\} . \qquad (37)$$

Solving the conditions in (37) we can express $\mathcal{L}$ in parametric form as

$$(\lambda, y, z) = \mathcal{L}(\xi_1) := \left(u, \alpha(\lambda - 1)^2, -\alpha(\lambda + 1)^2 \right) . \qquad (38)$$

This gives us the first result as follows.

Lemma 4. *The non-hyperbolic set* $\mathcal{L}$ *is transverse to the fast direction of (35).*

Proof. By differentiating (38) with respect to λ, we find that the curve $\mathcal{L}$ has tangent vector $e_{\mathcal{L}} = (1 \, , \, 2\alpha(\lambda - 1) \, , \, -2\alpha(\lambda + 1))$, which for all $|\lambda| < 1$ is transverse to the coordinate axes provided $\alpha \neq 0$. $\qquad \square$

While the non-hyperbolic curve $\mathcal{L}$ is now in a general position with respect to the fast variable, generically there may exist a new singularity along $\mathcal{L}$, where the flow's projection along the λ-direction onto the nullcline $f = 0$ is indeterminate. We shall call this the *star* singularity, defined in the following lemma.

Lemma 5. *For the values of the constants a_1, a_2, b_1, b_2, given in Proposition 3, there exists an isolated singularity of the flow along the non-hyperbolic set $\mathcal{L}$, where the projection of the slow flow onto $\mathcal{M}$ lies tangent to $\mathcal{L}$.*

Proof. Let us consider the slow critical subsystem (i.e. the sliding system (4)) obtained by letting $\varepsilon = 0$ in (35),

$$(0, \dot{y}, \dot{z}) \; = \; (f(0, y, z; \lambda), F_2(0, y, z; \lambda), F_3(0, y, z; \lambda)) \; . \tag{39}$$

$\mathcal{M}$ is the surface where $f = 0$, so a solution of (39) that remains on $\mathcal{M}$ for an interval of time satisfies $\dot{f} = 0$. We can find $\dot{\lambda}$ on $\mathcal{M}$ using the chain rule, writing $\dot{f} = \left(\dot{\lambda}, \dot{y}, \dot{z}\right) \cdot (\partial/\partial\lambda, \partial/\partial y, \partial/\partial z) f = \dot{\lambda}\frac{\partial f}{\partial\lambda} + (F_2, F_3) \cdot \frac{\partial f}{\partial(y,z)} = 0$, which rearranges to $\dot{\lambda} = -(F_2, F_3) \cdot \frac{\partial f}{\partial(y,z)} / \left(\frac{\partial f}{\partial\lambda}\right)$. Thus $\dot{\lambda}$ is indeterminate on $\mathcal{M}$ at points where the numerator and denominator of this vanish, or in full, where

$$0 = f = \frac{\partial f}{\partial \lambda} = (F_2, F_3) \cdot \frac{\partial f}{\partial(y, z)} \; . \tag{40}$$

These three conditions define an isolated singularity on $\mathcal{L} \subset \mathcal{M}$. Denoting the value of F_i at the singularity as F_{is}, and solving (40), we must find λ_s such that

$$\begin{aligned}
0 &= \tfrac{1}{2}(F_{2s}, F_{3s}) \cdot (-1 - \lambda_s, 1 - \lambda_s) \\
&= \left(\tfrac{a_1 + b_2}{2} + \tfrac{a_1 - b_2}{2}\lambda_s \; , \; \tfrac{b_1 + a_2}{2} + \tfrac{b_1 - a_2}{2}\lambda_s \right) \cdot \left(-\tfrac{1 + \lambda_s}{2}, \tfrac{1 - \lambda_s}{2} \right) \; ,
\end{aligned} \tag{41}$$

and find that the 'star' singularity lies at $(\lambda, y, z) = (\lambda_s, x_{2s}, x_{3s})$, where

$$\lambda_s = \frac{-\frac{a_1 + a_2}{b_1 - b_2} \pm \sqrt{1 + \frac{4a_1 a_2}{(b_1 - b_2)^2}}}{1 + \frac{a_1 - a_2}{b_1 - b_2}} \; , \quad x_{2s} = \alpha(\lambda_s - 1)^2 \; , \quad x_{3s} = -\alpha(\lambda_s + 1)^2 \; . \tag{42}$$

Noting that a_1 and a_2 in the normal form just take values ± 1, we have:

- in the case $a_1 = a_2 = 1$, we have $\lambda_s = \frac{-2}{b_1 - b_2} \pm \sqrt{1 + \frac{4}{(b_1 - b_2)^2}}$, implying that there exists a unique solution $\lambda_s \in (-1, +1)$ for any b_1 and b_2 (the positive root for $b_1 > b_2$, the negative root for $b_1 < b_2$);

- in the case $a_1 = a_2 = -1$, we have $\lambda_s = \frac{2}{b_1 - b_2} \pm \sqrt{1 + \frac{4}{(b_1 - b_2)^2}}$, implying that there exists a unique solution $\lambda_s \in (-1, +1)$ for any b_1 and b_2 (the positive root for $b_1 < b_2$, the negative root for $b_1 > b_2$);

- in the case $a_1 = -a_2 = 1$, we have $\lambda_s = \pm\sqrt{\frac{b_1 - b_2 - 2}{b_1 - b_2 + 2}}$, implying that there exist two solutions $\lambda_s \in (-1, +1)$ for $b_1 - b_2 > 2$, and no points otherwise.

- in the case $a_1 = -a_2 = -1$, we have $\lambda_s = \pm\sqrt{\frac{b_1 - b_2 + 2}{b_1 - b_2 - 2}}$, implying that there exist two solutions $\lambda_s \in (-1, +1)$ for $b_1 - b_2 < -2$, and no points otherwise.

$\square$

This lemma establishes the existence of at least one unique folded singularity on $\mathcal{L}$ in the cases listed in Proposition 3. In the cases where λ_s is unique we proceed directly to the steps that follow below. In the cases where λ_s can take two values we can proceed with the following analysis about each value, and will obtain different constants in the final local expression, i.e. a different folded singularity corresponding to each λ_s. In the cases when λ_s does not exist, no equivalence can be formed; these are the cases when the two-fold's sliding portrait is of focal type (see [2]), and there exists no orbits passing directly between the attracting and repelling branches of sliding, since orbits wind around the two-fold but never enter or leave it. So excluding those cases $a_1 = -a_2 = 1$ with $b_1 - b_2 \le 2$ and $a_1 = -a_2 = -1$ with $b_1 - b_2 \ge -2$, we proceed with the final step in proving proposition 3.

Lemma 6. *Coordinates can be defined in which the folded singularity of (35) lies at the origin, and $\mathcal{L}$ lies along a coordinate axis corresponding to a slow variable.*

Proof. Taking a valid solution of λ_s from (42) for $|\lambda| < 1$, a translation puts the singularity at the origin of the new coordinates

$$\xi_1 = \lambda - \lambda_s , \qquad \xi_2 = y - x_{2s} , \qquad \xi_3 = z - x_{3s} . \tag{43}$$

Then f becomes

$$f = -\frac{1 + \lambda_s}{2}\xi_2 + \frac{1 - \lambda_s}{2}\xi_3 - \left(\frac{\xi_3 + \xi_2}{2} + \alpha\xi_1\right)\xi_1 , \tag{44}$$

found by using (41)-(42) to ensure that terms involving y_s and z_s vanish. To find coordinates in which $\mathcal{L}$ lies along a coordinate axis, from (38) we can obtain the ξ_1-parameterized expression for $\mathcal{L}$,

$$(\xi_1, \xi_2, \xi_3) = \mathcal{L}(\xi_1) := (\xi_1, -\alpha\xi_1(2 - 2\lambda_s - \xi_1), -\alpha\xi_1(2 + 2\lambda_s + \xi_1)) ,$$

and re-arrange this to take ξ_3 as a parameter, expressing $\mathcal{L}$ as $(\xi_1, \xi_2) = (\xi_{1L}(\xi_3), \xi_{2L}(\xi_3))$, where

$$\begin{pmatrix} \xi_{1L}(\xi_3) \\ \xi_{2L}(\xi_3) \end{pmatrix} := \begin{pmatrix} -1 - \lambda_s + \sqrt{(1 + \lambda_s)^2 - \xi_3/\alpha} \\ -\xi_3 - 4\alpha(-1 - \lambda_s + \sqrt{(1 + \lambda_s)^2 - \xi_3/\alpha}) \end{pmatrix} . \tag{45}$$

The derivatives of these functions are needed to evaluate the vector field components below, these are

$$\xi'_{1L}(\xi_3) = \frac{-1/2\alpha}{1 + \lambda_s + \xi_{1L}(\xi_3)} \ , \qquad \xi'_{2L}(\xi_3) = \frac{1 - \lambda_s - \xi_{1L}(\xi_3)}{1 + \lambda_s + \xi_{1L}(\xi_3)} \ . \tag{46}$$

We can then rectify $\mathcal{L}$ to lie along some ζ_3 axis by defining new coordinates

$$\zeta_1 = \xi_1 - \xi_{1L}(\xi_3) \ , \qquad \zeta_2 = \xi_2 - \xi_{2L}(\xi_3) \ , \qquad \zeta_3 = \xi_3 \ . \tag{47}$$

The original vector field components can then be written as

$$\begin{aligned}
f &= -\tfrac{1+\lambda_s+\xi_{1L}}{2}\zeta_2 - \alpha\zeta_1^2 - \zeta_2\zeta_1/2 \ , \\
F_2 &= F_{2s} + (\zeta_1 + \xi_{1L}(\zeta_3))\tfrac{\partial F_{2s}}{\partial \phi_1} = F_{2s} + \mathsf{O}\left(\zeta_1, \zeta_3\right) \ , \\
F_3 &= F_{3s} + (\zeta_1 + \xi_{1L}(\zeta_3))\tfrac{\partial F_{3s}}{\partial \phi_1} = F_{3s} + \mathsf{O}\left(\zeta_1, \zeta_3\right) \ .
\end{aligned} \tag{48}$$

With a little algebra we find that

$$\begin{aligned}
\varepsilon\dot{\zeta}_1 &= \varepsilon\dot{\xi}_1 - \varepsilon\dot{\xi}_3\xi'_{1L}(\xi_3) \\
&= \frac{1+\lambda_s}{2}\left(\frac{\varepsilon F_{3s}}{\alpha(1+\lambda_s)^2} - \zeta_2\right) - \alpha\zeta_1^2 + \mathsf{O}\left(\varepsilon\zeta_3, \varepsilon\zeta_1, \zeta_2\zeta_3, \zeta_2\zeta_1, \zeta_1^3\right)
\end{aligned}$$

A small shift $\tilde{\zeta}_2 = \zeta_2 - \frac{\varepsilon F_{3s}}{\alpha(1+\lambda_s)^2}$ yields, after some lengthy but straightforward algebra, using the relations in (41) and (48) to show that any terms not proportional to ζ_1 or ζ_3 vanish,

$$\dot{\tilde{\zeta}}_2 = F_2 - \dot{\zeta}_3\xi'_{2L}(\zeta_3) = q\zeta_1 + \frac{\tilde{p}}{\alpha}\zeta_3 + \mathsf{O}\left(\zeta_3^2, \zeta_1\zeta_3\right)$$

where

$$\tilde{q} = \frac{\partial F_{2s}}{\partial\lambda} - \frac{\partial F_{3s}}{\partial\lambda}\frac{1-\lambda_s}{1+\lambda_s} \ , \qquad \tilde{p} = -\frac{2F_{3s} + \tilde{q}(1+\lambda_s)^2}{2(1+\lambda_s)^2} \ .$$

The last thing to do is just scaling. Collecting everything together so far we have

$$\begin{aligned}
\varepsilon\dot{\zeta}_1 &= (d_1\tilde{\zeta}_2 - \alpha\zeta_1^2)\lambda_s + \mathsf{O}\left(\varepsilon z, \varepsilon\zeta_3, \zeta_1\zeta_3\right) \\
\dot{\tilde{\zeta}}_2 &= \tfrac{\tilde{p}}{\alpha}\zeta_3 + \tilde{q}\zeta_1 + \mathsf{O}\left(\zeta_3^2, \zeta_1\zeta_3\right) \\
\dot{\zeta}_3 &= F_{3s} + \mathsf{O}\left(\zeta_3, \zeta_1\right)
\end{aligned}$$

where $d_1 = -\frac{1}{2}(1+\lambda_s)$. Defining new variables $\eta_1 = \sqrt{|\alpha|}\zeta_1$, $\eta_2 = -\text{sign}(\alpha)d_1\tilde{\zeta}_2$, $\eta_3 = -\text{sign}(\alpha)\zeta_3$, and $\tilde{t} = -\text{sign}(\alpha)t$, gives

$$\begin{aligned}
\varepsilon\dot{\eta}_1 &= \eta_2 + \eta_1^2 + \mathsf{O}\left(\varepsilon\eta_1, \varepsilon\eta_3, \eta_1\eta_3\right) \\
\dot{\eta}_2 &= p\eta_3 + q\eta_1 + \mathsf{O}\left(\eta_3^2, \eta_1\eta_3\right) \\
\dot{\eta}_3 &= r + \mathsf{O}\left(\eta_3, \eta_1\right)
\end{aligned} \tag{49}$$

where

$$r = F_{3s} \,, \quad p = -\tfrac{1}{4|\alpha|}\left(F_{2s} + F_{3s} - 2q\sqrt{|\alpha|}\right) \,,$$
$$q = \tfrac{-1}{2\sqrt{|\alpha|}}\left((\lambda_s + 1)\tfrac{\partial F_{2s}}{\partial \lambda} + (\lambda_s - 1)\tfrac{\partial F_{3s}}{\partial \lambda}\right) \,.$$

(50)

Replacing (η_1, η_2, η_3) with (x, y, z), this is the result in the lemma and in Proposition 6, clearly valid only for $\alpha \neq 0$ (otherwise the transformation is singular). $\qquad \square$

Figure 6 shows an example of the perturbed system and its switching layer for each flavour of two-fold in (i) (corresponding to those in figure 4), followed by their switching layer (ii), and a rotation (iii) to show the phase portrait around the set $\mathcal{L}$ more clearly (similar to figure 5). In the mixed visible-invisible two-fold (far right column), for example, the nonsmooth system (i) has a phase portrait with infinitely many intersecting trajectories traversing the singularity, while the layer system (ii-iv) splits these into distinguishable orbits, a finite number of which asymptote to the attracting and repelling branches of the critical manifold.

Like the different kinds of two-fold, there are different classes of the 'star' singularity, and their classification depends on the dynamics inside $\mathcal{M}$ (on the t timescale). From the expression (49) with (50) we see that the class therefore depends not only on the constants a_1, a_2, b_1, b_2, of the original piecewise smooth system, but also on the 'hidden' parameter α.

The classification scheme is fairly simple, and can be used to verify the dynamics on $\mathcal{M}$ seen in figure 6. The projection of the system (49) onto $\mathcal{M}$, found by differentiating the condition $0 = \eta_2 + \eta_1^2$ with respect to time to give $0 = b\eta_3 + c\eta_1 + 2\eta_1\dot{\eta}_1 + \mathsf{O}\left(\eta_3^2, \eta_1\eta_3\right)$, is

$$\begin{pmatrix} \dot{\eta}_1 \\ \dot{\eta}_3 \end{pmatrix} = \frac{1}{-2\eta_1}\left\{\begin{pmatrix} c & b \\ -2a & 0 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_3 \end{pmatrix} + \mathsf{O}\left(\eta_3^2, \eta_1\eta_3\right)\right\} \,.$$

A classification then follows by neglecting the singular pre-factor $1/2\eta_1$ and considering whether the phase portrait is that of a focus, a node, or a saddle. This is determined by the 2×2 matrix Jacobian, which has trace c, determinant $2ab$, and eigenvalues $\frac{1}{2}(c \pm \sqrt{c^2 - 8ab})$. This will not be the true system's phase portrait because the time-scaling from the $1/2x$ factor is positive in the attractive branch of $\mathcal{M}$, negative (time-reversing) in the repulsive branch, and divergent at the singularity (turning infinite time convergence to the singularity into finite time passage *through* the singularity). The effect of this is to 'fold' together attracting and repelling pairs of each equilibrium type, so each equilibrium becomes a 'folded-equilibrium', forming a continuous bridge between branches of $\mathcal{M}$.

As a result the flow on $\mathcal{M}$ is a folded-saddle if $ab < 0$, a folded-node if $0 < 8ab < c^2$, and a folded-focus if $c^2 < 8ab$. So-called canard cases occur for $c > 0$ and faux canard for $c < 0$ (a
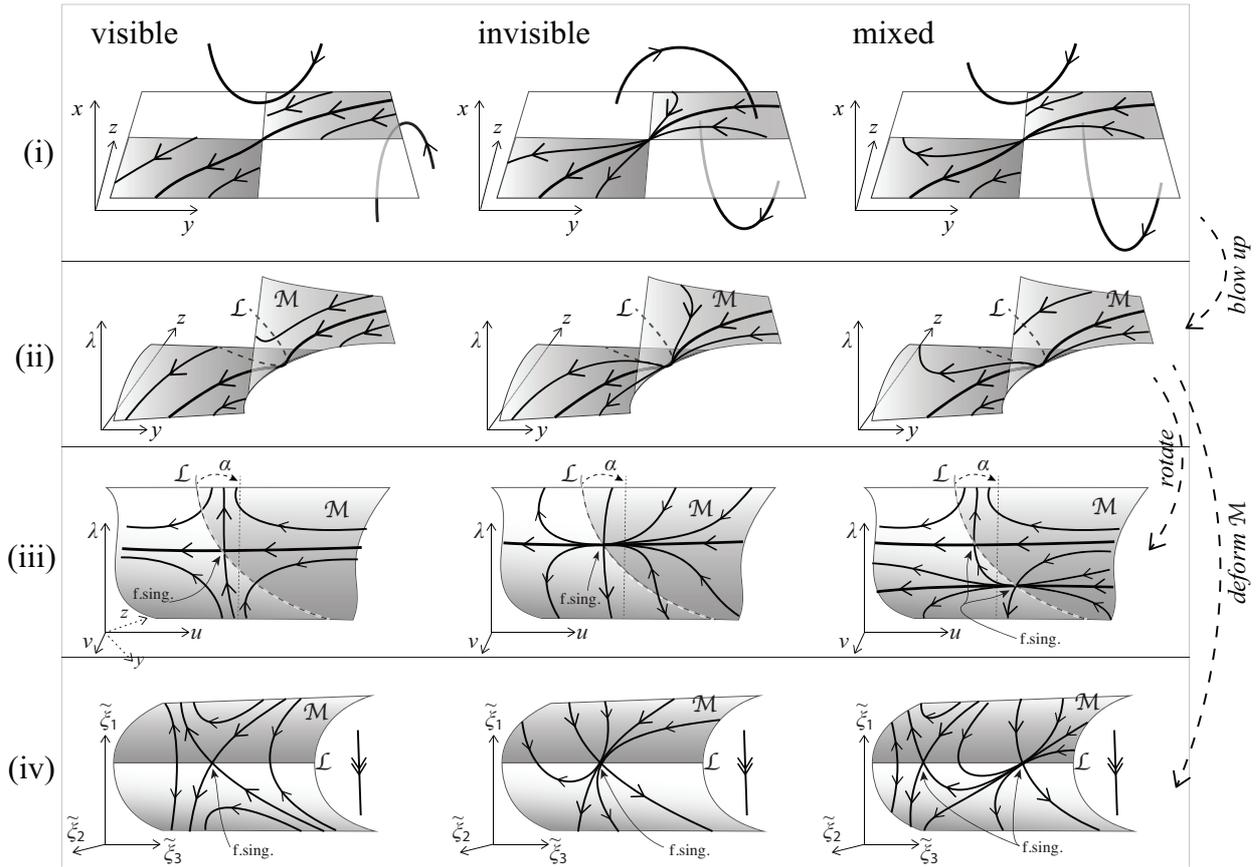
FIG. 6: blowing up the perturbed ($\alpha \neq 0$) system, for examples of each flavour of two-fold. Labelling as in figure 5. Note in the regularization (ii) that $\mathcal{L}$ is now a curve. Rotating around the u axis in (iii) we can see the attracting branch (upper right segment) and repelling branch (lower left segment) of the sliding manifold $\mathcal{M}$ (shaded), connected by $\mathcal{L}$. The folded singularity (f.sing.) appears along $\mathcal{L}$, two in the case of mixed visibility, recognised as having a phase portrait that resembles a saddle or node if we reverse time in the repelling branch of $\mathcal{M}$. In (iv) we sketch the corresponding phase portraits in the slow-fast system (49).

canard is a trajectory that evolves directly from the attracting sliding region to the repelling sliding region, while a faux canard does the opposite). In the visible two-fold the singularity becomes a folded-saddle, in the invisible case it becomes a folded-node, while the mixed case becomes a pair consisting of one folded-saddle and one folded-node.

In the cases depicted in figures 4-6, there exist one or more trajectories, passing from the attractive sliding region to the repelling sliding region. This passage occurs in finite time (since the vector field obtained by substituting (29) into (2) is non-vanishing everywhere locally). The flow is unique in forward time everywhere except in the repelling sliding region, where it is set-valued because trajectories may slide along $x = 0$, but may also be ejected into $x > 0$ or $x < 0$ at

any point. This means that the flow may evolve deterministically until it arrives at the singularity, at which point it becomes set-valued, so we say that determinacy breaking occurs at the singularity whenever the sliding flow passes from the attractive to repelling sliding region. This is illustrated in figure 7. It occurs in the invisible case when $b_1, b_2 < 0$ and $b_1 b_2 > 1$, in the visible case when $b_1 < 0$ or $b_2 < 0$ or $b_1 b_2 < 1$, and finally in the mixed case when $b_1 < 0 < b_2$ and $b_1 b_2 < -1$ or when $b_1 + b_2 < 0$ and $b_1 - b_2 < -2$. (The particular cases shown in figure 7 are: (i) $a_1 = a_2 = -1$ with $b_1 < 0$ or $b_2 < 0$ or $b_1 b_2 < 1$; (ii) $a_1 = a_2 = 1$ with $b_1, b_2 < 0$ and $b_1 b_2 > 1$; (iii) $a_1 a_2 = -1$ with $b_1 < 0 < b_2$ and $b_1 b_2 < -1$ or with $b_1 + b_2 < 0$ and $b_1 - b_2 < -2$.) The phase portraits in figure 6 resolve this passage through the singularity in more detail, revealing strong and weak eigendirections, and showing that the determinacy breaking trajectories persist in the switching layer.
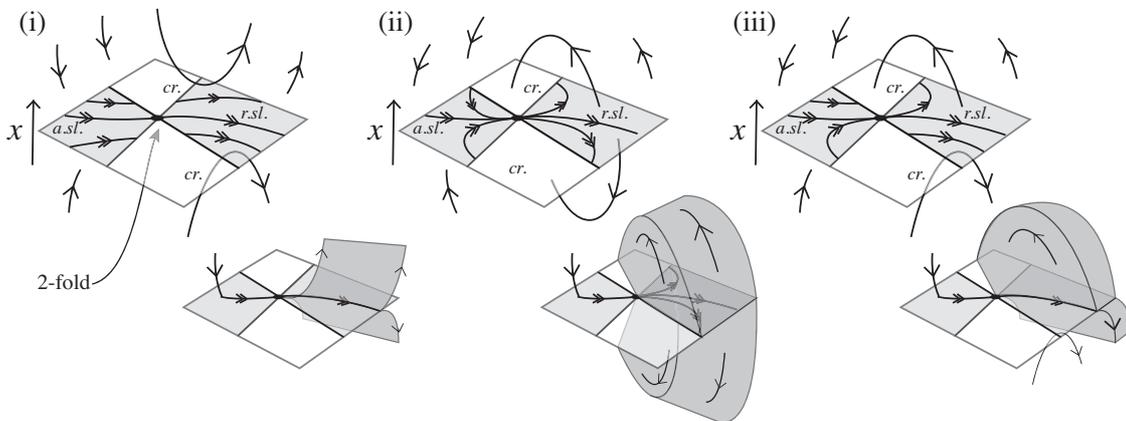
FIG. 7: Three kinds of two-fold, showing determinacy-breaking in the systems from figure 4, meaning that the flow becomes set-valued at is passes through the singularity. The set has 2 dimensions in (i) and 3 dimensions in (ii-iii).

In the literature on smooth two timescale systems, the connection of attracting and repelling branches of a slow invariant manifolds has been well studied, leading to a generic canonical form (49) as described in [32]. In the present notation, this requires $\mathcal{M}$ and $\mathcal{L}$ to be non-degenerate, given by

$$f = 0 \, , \qquad \frac{\partial f}{\partial \lambda} = 0 \, , \qquad \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \neq 0 \, , \qquad \frac{\partial^2 f}{\partial \lambda^2} \neq 0 \, , \tag{51}$$

the first three of which are satisfied on $\mathcal{L}$ as given by (9), while the fourth holds only for $\alpha \neq 0$. A 'star' singularity is a generic point along $\mathcal{L}$ where moreover $0 = f_1 = \dot{f}_1 = \frac{\partial f_1}{\partial x}$, with non-degeneracy conditions $\frac{\partial g_1}{\partial y}, \frac{\partial g_1}{\partial z} \neq 0 \neq \frac{\partial^2 g_1}{\partial x^2}$.

To avoid confusion we have referred to this as the 'star' singularity. In smooth two timescale

systems it is usually known as a 'folded' singularity (a folded node, folded saddle, etc. depending on the phase portrait inside $\mathcal{M}$) [32], an unfortunate clash of 'fold' terminologies with the piecewise smooth literature, rooted in much older work on singularities involving flows and surfaces. A qualitative association between the two-fold and the star singularity was made in [3, 4], based on similarities between the phase portraits on $\mathcal{M}$, and the result above at last clarifies this relation, but requires obtaining structural stability inside the switching layer via introduction of terms nonlinear in λ.

One may ask why certain cases were excluded by the proposition above. The excluded cases were those in which no canards or faux canards exist in the slow-fast system. Trajectories connecting the attractive and repulsive sliding regions occur when transversal intersections exist between the attracting and repelling branches of the sliding manifolds $\mathcal{M}$. If no such intersections exist, the critical system possesses no 'star' singularities and hence is excluded from Proposition 3. Hence the omission of these cases is consistent, and a posteriori it is obviously necessary in the equivalence sought in the proposition.

This just touches the surface of the problem, opening the way to deeper study of the two-fold's dynamics. A glance at the papers [5, 32, 33] reveals what happens if we smooth out the discontinuity, for reasons of regularization analysis or numerical simulation. As regularization has been a popular method for studying the extent to which smooth and discontinuous systems approximate each other, we clarify the relation between switching layer dynamics and regularization in the next section. A detailed description of the star singularity in smooth systems can be found in the references above. Its implications for the piecewise smooth system are of interest only in the singular limit $\varepsilon = 0$, not, as concerns singular perturbation studies, what happens for $\varepsilon > 0$. The full consequences for the larger piecewise smooth system, too lengthy to begin here, will be pursued in future work.

Given the continued interest in the two-fold after many years, we should discuss also how the results above fit into the growing literature on the subject. We delay this to closing comments in section VII.

VI. REGULARIZATION

The piecewise smooth system is sometimes compared to its regularization, the smooth system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}; \phi_\kappa(h)) = \frac{1 + \phi_\kappa(h)}{2} \mathbf{f}^+(\mathbf{x}) + \frac{1 - \phi_\kappa(h)}{2} \mathbf{f}^-(\mathbf{x}) , \tag{52}$$

in terms of a smooth sigmoid function

$$\phi_\kappa(h) \in \begin{cases} \text{sign}\, h & \text{if } |h| \geq \kappa \,, \\ (-1, +1) & \text{if } |h| < \kappa \,, \end{cases} \tag{53}$$

for which $\phi'_\kappa(h) > 0$ for $|h| < \kappa$, with $\phi_\kappa(h) \to \text{sign}\, h$ as $\kappa \to 0$. Taking coordinates in which $h = x$ as in section II, let $u = x/\kappa$ and note that $\phi_\kappa(x) = \phi_\kappa(\kappa u) = \phi_1(u)$, then for κ zero (52) becomes the differential-algebraic system

$$\begin{aligned} 0 &= f(0, X; \phi_1(u)) \,, \\ \dot{X} &= F(0, X; \phi_1(u)) \,, \end{aligned} \tag{54}$$

This is just the sliding problem (4) rewritten with λ replaced by $\phi_\kappa(x)$ in the limit $\kappa \to 0$. It is also the singular limit of the smooth system (52), which written in terms of u is

$$\begin{aligned} \kappa \dot{u} &= f(\kappa u, X; \phi_1(u)) \,, \\ \dot{X} &= F(\kappa u, X; \phi_1(u)) \,. \end{aligned} \tag{55}$$

The system (55) provides a smoothed two-timescale system approximating the flow near the discontinuity surface $x = 0$ of the original discontinuous problem (1). Below we show that this is qualitatively consistent with the switching layer system (5), and moreover that they are equivalent in the limit $\kappa \to 0$, $\varepsilon \to 0$.

From (55) we proceed analogously to section II. The outcome that the algebraic condition in (55) defines a hypersurface

$$\mathcal{M}' = \{(u, X) \in \mathbb{R}^n \,:\, 0 = f(0, X; \phi_1(u))\} \,, \tag{56}$$

which is the critical manifold of (54) (the invariant manifold of (54)) wherever it is normally hyperbolic, that is, excepting a set of points

$$\mathcal{L}' = \{(u, X) \in \mathcal{M}' \,:\, f_{,u}(0, X; \phi_1(u)) = 0\} \,. \tag{57}$$

The degeneracy of this set follows similarly to $\mathcal{L}$ in (9), because all higher derivatives of $f(0, X; \phi_1(u))$ with respect to u vanish, but this fact is less obvious than the vanishing of derivatives of $f(0, X; \lambda)$ with respect to λ in section II. The second derivative is

$$f_{,uu} = \frac{\phi''_1(u)}{\phi'_1(u)} f_{,u} + (\phi'_1(u))^2 f_{,\phi_1 \phi_1} \,. \tag{58}$$

The first term vanishes on $\mathcal{L}'$ where $f_{,u} = 0$, and the second vanishes for (52) as $f_{,\phi_1 \phi_1} = 0$.

This means that expanding the first line of (5) as a series in u about some $u_0 \in (-\kappa, +\kappa)$,

$$f(0, X; \phi_1(u)) = f(0, X; \phi_1(u_0)) + (u - u_0)f_{,u}(0, X; \phi_1(u_0)) + \mathsf{O}\left((u - u_0)^2\right) . \tag{59}$$

On the set $\mathcal{L}'$ the two lowest order terms in this expression vanish, leaving

$$f(0, X; \phi_1(u)) = \mathsf{O}\left((u - u_0)^2\right) , \tag{60}$$

which vanishes identically for $\mathbf{f}$ as defined in (52), so the first line of (54) or (55) is trivial on $\mathcal{L}'$, and the value of u that defines $\mathcal{L}'$ is undetermined. Geometrically this means that $\mathcal{L}'$ lies along the u coordinate direction, and the fast direction of the corresponding perturbed system (55).

If we allow terms nonlinear in λ to be added to (2), in the form (14), and only then regularize by substituting $\lambda = \phi_\kappa(h)$, as in section III we obtain a topologically distinct system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}; \phi_\kappa(h)) = \frac{1 + \phi_\kappa(h)}{2}\mathbf{f}^+(\mathbf{x}) + \frac{1 - \phi_\kappa(h)}{2}\mathbf{f}^-(\mathbf{x}) + (1 - \phi_\kappa^2(h))\mathbf{g}(\mathbf{x}; \phi_\kappa(h)) , \tag{61}$$

that does not exhibit a degeneracy of $\mathcal{L}'$ under generic conditions.

The Sotomayor-Teixeira theory of regularization [26] can be used to show the conjugacy between the two timescale dynamics of (52) in the limit of small κ, and in [22] this theory was extended to apply to systems (61).

Having shown that the switching layer approach of section II and the regularization approach above exhibit similar degeneracies, let us show that their critical dynamics is in fact equivalent in the limit of small ε or κ.

Proposition 7. *Letting $\lambda = \phi_\kappa(x)$ given $\dot{x} = f(\mathbf{x}; \lambda)$, with ϕ_κ defined in (53), the dynamics of λ is given by*

$$\varepsilon(\lambda, \kappa)\dot{\lambda} = \dot{x} = f(\mathbf{x}, \lambda) \tag{62}$$

such that $\varepsilon(\lambda, \kappa) \ll 1$, where ε denotes a continuous positive function and κ a small parameter, with $0 < \kappa < \kappa^* \ll 1$ for $\lambda \in (\phi_{\kappa*}^-, \phi_{\kappa*}^+)$, in terms of constants κ^* and $\phi_{\kappa*}^\pm$ that satisfy $\phi_{\kappa*}^\pm \to \pm 1$ as $\kappa^* \to 0$.

Proof. We shall use the relation $\lambda = \phi_\kappa(x)$ to derive a dynamical system on λ for $|x| < \kappa$. Differentiating $\lambda = \phi_\kappa(x)$ with respect to t gives

$$\dot{\lambda} = \frac{\dot{x}}{\kappa}\phi_\kappa'(x) \qquad \text{for} \quad |x| < \kappa . \tag{63}$$

Considering a variable $u = x/\kappa$ we see from (53) that $\phi_\kappa(x) = \phi_\kappa(u\kappa) = \phi_1(u)$, with derivative $\kappa\phi_\kappa'(x) = \phi_1'(u)$. Both $\phi_1(u)$ and $\phi_1'(u)$ are smooth with respect to u and independent of κ.

Moreover $\phi_1'(u)$ is strictly positive because $\phi_1(u)$ is strictly increasing, and $\phi_1'(x/\kappa)$ only becomes small (or vanishing) for $|x|/\kappa > 1$. So the quantity κ/ϕ_1' is small and nonzero for $|x|/\kappa \le 1$, and using it we define a fast timescale $\tau = t\phi_\kappa'(x)/\kappa$. Since $\phi_1(u)$ is differentiable and monotonic for $|u| < 1$, it has an inverse $\psi(\lambda)$ such that $\psi(\phi_1(u)) = u$, and we can define a function

$$\varepsilon(\lambda, \kappa) := \kappa/\phi_1'(\psi(\lambda)) , \qquad \text{for} \quad |\lambda| < 1 . \tag{64}$$

That this quantity is small is shown as follows: the function $\phi_1(u)$ varies differentiably over an interval on which its extremal values are $\phi_1(\pm 1) = \pm 1$, therefore there exists a point u_* where $\phi'(u_*) = \frac{\phi_1(+1) - \phi_1(-1)}{(+1) - (-1)} = 1$, and by continuity since $\phi_1'(\pm 1) = 0$, there exist two points $u_\kappa^\pm$ where $\phi'(u_\kappa^\pm) = \pm\kappa$ for $0 < \kappa < 1$, and moreover an interval $u_\kappa^- < u < u_\kappa^+$ such that $\phi_1'(u) > \kappa$. Fix some κ_* such that $0 < \kappa_* \ll 1$, then $\kappa/\phi_1(u) < 1$ for $u_{\kappa*}^- < u < u_{\kappa*}^+$, and

$$\lim_{\kappa \to 0} \varepsilon(\lambda, \kappa) = 0 ,$$

so that $\varepsilon(\lambda, \kappa) \ll 1$ for $\kappa \ll \kappa_*$ and $u \in (u_{\kappa*}^-, u_{\kappa*}^+)$.

By (63) we therefore have the dynamical equation $\varepsilon(\lambda, \kappa)\dot\lambda = \dot x = f(\mathbf{x}, \lambda)$ for small ε in the proposition. $\qquad\square$

This proposition identifies λ as a fast variable inside $\lambda \in (-1, +1)$ (more strictly for $\lambda \in (\phi_{\kappa*}^-, \phi_{\kappa*}^+)$ where $\phi_{\kappa*}^\pm = \phi_1(u_{\kappa*}^\pm)$, and κ^* is arbitrarily small but nonzero). When λ is set-valued on $x = 0$ with $\kappa = 0$, this equation determines the variation of λ on the timescale τ which is instantaneous relative to the timescale t.

The two timescale system obtained by combining (62) with (52) on the interval $\lambda \in (\phi_{\kappa*}^-, \phi_{\kappa*}^+)$, is then

$$(\varepsilon\dot\lambda, \dot X) = (f(\kappa u, X; \lambda), \ F(\kappa u, X; \lambda)) = (f(0, X; \lambda), \ F(0, X; \lambda)) + \mathsf{O}(\kappa u) . \tag{65}$$

The truncation of this is formally the same as the switching layer system (5), i.e.

$$(\varepsilon\dot\lambda, \dot X) = (f(0, X; \lambda), \ F(0, X; \lambda)) . \tag{66}$$

While standard geometrical singular perturbation theory does not apply to (65)-(66) when derived in this way from the smoothing (52), because ε is then a function rather than a parameter, we can show that (5) has the same critical manifold geometry as the regularization (55). We can omit the arguments of $\varepsilon = \varepsilon(\lambda, \kappa)$ without loss of generality.

Proposition 8. *The system (5) has equivalent slow-fast dynamics to the system (52) on the discontinuity set $x = 0$ in the critical limit $\varepsilon = 0$.*

Proof. Rescaling time in (5) to $\tau = t/\varepsilon$, then setting $\varepsilon = 0$ and $x = 0$, gives the fast critical subsystem

$$(\lambda', X') = (f(0, X; \lambda), \ 0) \ . \tag{67}$$

The equilibria of this one-dimensional system form the manifold $\mathcal{M}$ defined in (8), which is equivalent to the manifold $\mathcal{M}'$ given by (56) for $\lambda = \phi_1(u)$. This is an invariant manifold of the system (67) in the $\varepsilon = 0$ limit everywhere that $\mathcal{M}$ is normally hyperbolic, that is excepting the set $\mathcal{L}$ defined in (9). Since $f_{,u} = \phi_1'(u)f_{,\lambda}$ and $\phi_1'(u) \neq 0$ for $|u| < 1$, this definition of $\mathcal{L}$ is equivalent to $\mathcal{L}'$ given by (57). Setting $\varepsilon = 0$ and $x = 0$ in (5) gives the slow critical subsystem (4),

$$(0, \dot{X}) = (f(0, X; \lambda), \ F(0, X; \lambda)) \ ,$$

which defines dynamics in the critical limit $\varepsilon = 0$ on $\mathcal{M}$, which is identical to (54). $\qquad\square$

Since the dynamics of (5) is of interest only in the limit $\varepsilon = 0$, where the dynamics of λ is infinitely fast, further study away from this limit is beyond the interest of piecewise smooth dynamics itself. Evidently there is more to be analysed in the relation between (5) and (65) for κ and ε nonzero as a more general singular perturbation problem, concerning how closely smooth systems can approximate discontinuous models or vice versa.

VII.   CLOSING REMARKS

The two-fold singularity is of particular interest where degeneracies in the switching layer are concerned. Of all elementary singularities in piecewise smooth flows, the two-fold has proven surprisingly difficult to characterise, from its first description in [10, 27] in three dimensions, to its study in higher dimensions in [2]. These (and all references in between to the author's knowledge) exclusively consider the class of *Filippov system* obtained by placing (29) into the convex combination (2). The degeneracy raised in this paper adds a new level of intricacy.

Early questions about the structural stability of two-fold singularities, raised in [27] particularly, have been largely resolved by uncovering the intricate crossing and sliding phase portraits, revealing various topologically stable classes separated by bifurcations [2, 9, 14], which include the birth of limit cycles, bifurcation of an invariant nonsmooth diabolo, and passage of sliding equilibria through the singularity. It is important to note that away from the known bifurcations that affect it, the two-fold is neither an attractor nor repeller, so the flow either misses the singularity, or traverses it in finite time.

Particularly because of some similarity to canard dynamics and issues of flow uniqueness as discussed in section V, attention has turned to how the two-fold can be understood as a limit or approximation of a smooth flow. An equivalence between "sliding" motion along a discontinuity surface, and "slow" motion on invariant manifolds of a smooth two-timescale system, has been shown [22, 26], and the similarities in their respective behaviours continue to be of interest [3, 4, 20, 29]. The papers [17, 20, 29] follow the route of substituting (29) into (2), and then regularizing, and therefore exhibit a degenerate sliding manifold as described in section II. While the degeneracy is not noted explicitly, certain difficulties of robustness that arise from it are, and are tackled by re-scaling the local variables to prove that particular solutions called 'primary canards' persist within the Sotomayor-Teixeira regularization. Here we have shown instead that a more general regularization breaks the degeneracy, and in doing so we are able to relate the two-fold to the canonical form of generic non-hyperbolic curves along invariant manifolds in smooth two timescale systems.

The result in section II on the degeneracy of $\mathcal{L}$ extends to systems with multiple switches. Let the switching surface consist $r \leq n$ transversally intersecting manifolds $h_j(\mathbf{x}) = 0$ for $j = 1, ..., r$. We define switching multipliers $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_r)$, and instead of (2) we have (see e.g. [13])

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \boldsymbol{\lambda}) = \sum_{p_1,...p_r = \pm} \lambda_1^{(p_1)} ... \lambda_r^{(p_r)} \mathbf{f}^{p_1 ... p_r}(\mathbf{x}) \ , \qquad \lambda_i^{(\pm)} = \frac{1 \pm \lambda_i}{2} \ , \tag{68}$$

where $\mathbf{f}^{p_1 ... p_r}(\mathbf{x})$ denote 2^r different smooth vector fields $\mathbf{f}^{\pm \pm \cdots}(\mathbf{x})$, with

$$\begin{cases} \lambda_j = \text{sign} \, h_j & \text{if} \ \ h_j \neq 0 \ , \\ \lambda_j \in (-1, +1) & \text{if} \ \ h_j = 0 \ . \end{cases} \tag{69}$$

Take coordinates $\mathbf{x} = (x_1, x_2, ..., x_n)$ in which $x_j = h_j$ for $j = 1, ..., r$. For brevity write $X = (x_{r+1}, ..., x_n)$ so that $\mathbf{x} = (x_1, ..., x_r, X)$, and $F = (f_{r+1}, ..., f_n)$ so that $\mathbf{f} = (f_1, ..., f_r, F)$. Then at a point where r switching manifolds $h_j = 0$ intersect, the switching layer system on $x_1 = ... = x_r = 0$ is

$$\begin{aligned} \varepsilon_j \lambda_j' &= f_j(0, ..., 0, X; \lambda_1, ..., \lambda_r) \ , \qquad j = 1, ..., r \ , \\ X' &= F(0, ..., 0, X; \lambda_1, ..., \lambda_r) \ . \end{aligned} \tag{70}$$

The sliding manifold is given in by

$$\mathcal{M} = \left\{ (\lambda_1, ..., \lambda_r) \in (-1, +1)^r, \ X \in \mathbb{R}^{n-r} \ : \ \begin{array}{l} f_j(0, ..., 0, X) = 0, \\ \text{for} \ \ j = 1, ..., r. \end{array} \right\} \ , \tag{71}$$

and is normally hyperbolic except on a set

$$\mathcal{L} = \left\{ (\lambda_1, ..., \lambda_r, X) \in \mathcal{M} \ : \ \det \left| \frac{\partial(f_1, ..., f_r)}{\partial(\lambda_1, ..., \lambda_r)} \right| = 0 \right\} \ . \tag{72}$$

If we assume for some infinitesimal ε_0 that the ratios $\kappa_j = \varepsilon_j/\varepsilon_0$ are finite and nonzero as $\varepsilon_j \to 0$ and $\varepsilon_0 \to 0$, then with respect to the fast timescale $\tau = t/\varepsilon_0$, the layer system is

$$
\begin{aligned}
\lambda'_j &= f_j(0,...,0,X;\lambda_1,...,\lambda_r)/\kappa_j , \qquad j = 1,...,r , \\
X' &= \varepsilon_0 f_i(0,...,0,X;\lambda_1,...,\lambda_r) ,
\end{aligned}
\tag{73}
$$

with the prime denoting differentiation with respect to the fast time τ. This becomes an r dimensional fast subsystem in the limit $\varepsilon_0 \to 0$,

$$
\begin{aligned}
\lambda'_j &= f_j(0,...,0,X;\lambda_1,...,\lambda_r)/\kappa_j , \qquad j = 1,...,r , \\
X' &= 0 .
\end{aligned}
\tag{74}
$$

Expanding the first r line of (74) as a series in $\boldsymbol{\lambda} = (\lambda_1,...,\lambda_r)$ about some $\boldsymbol{\lambda}_* \in (-1,+1)^r$ on $\mathcal{M}$,

$$
f_j(0,...,0,X;\boldsymbol{\lambda}) = (\boldsymbol{\lambda} - \boldsymbol{\lambda}_*) \cdot \frac{\partial}{\partial \boldsymbol{\lambda}} f_j(0,...,0,X;\boldsymbol{\lambda}_*) + \mathsf{O}\left(|\boldsymbol{\lambda} - \boldsymbol{\lambda}_*|^2\right) .
\tag{75}
$$

On the set $\mathcal{L}$ the $r \times r$ square matrix $\frac{\partial(f_1,...,f_r)}{\partial(\lambda_1,...,\lambda_r)}$ is singular, so this expression is degenerate and unsolvable to lowest order. Hence $\boldsymbol{\lambda}$ is again not uniquely defined on $\mathcal{L}$, unless there exist terms of order $|\boldsymbol{\lambda} - \boldsymbol{\lambda}_*|^2$. In general, such terms may exist if there is at least multi-linear dependence on the λ_i's (terms like $\lambda_i\lambda_j$ for $i \neq j$), as is common for instance in genetic regulatory networks (see e.g. [21]).

Acknowledgements

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