Dynamics at a switching intersection: hierarchy, isonomy, and multiple-sliding

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Abstract

If a set of ordinary differential equations is discontinuous along some threshold, solutions can be found that are continuous, if sometimes multi-valued. We show the extent to which unique solutions can be found in general cases when the threshold takes the form of finitely many intersecting manifolds. If the intersections are transversal, finitely many solutions can be found that slide along the threshold. They are obtained by a hierarchical application of convex combinations to form a differential inclusion. The system chooses between these solutions by means of an instantaneous dummy system. No assumptions on attractivity are required and all switches are treated equally, so the standard 'Filippov' method is extended to intersections of discontinuity manifolds in the most natural way possible. The corresponding result in the setting of equivalent control is also given, allowing more general systems than typical linear control forms to be solved.

Keywords: Filippov, switching, sliding, discontinuity, dynamics

1. Introduction

An ordinary differential equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ has unique solution trajectories $\mathbf{x}(t)$ for given initial data if $\mathbf{f}$ is sufficiently smooth. If $\mathbf{f}$ is discontinuous along some hypersurface, then solution trajectories can be found by first interpolating between the values that $\mathbf{f}$ takes either side of the discontinuity, then admitting only values that give consistent dynamics. The difficult problem is to determine whether the resulting trajectories are unique, a problem that remains unresolved even in some relatively common scenarios. These problems are of interest because discontinuities are used to model switching of electrical relays or chemical channels in the living nervous system, the stick-slip motion of rigid bodies, and the effect of rule changes in

social models; a review can be found in a recent collection of articles [25].

A vital observation usually associated with Filippov [11, 23] is that in many cases a discontinuity can be resolved to give deterministic dynamics. The case where a discontinuity occurs at a smooth hypersurface is well understood. One obtains trajectories that either cross the discontinuity transversally or slide along it, as shown in figure 1. At a point p on the discontinuity manifold $\mathcal{D}$, one assumes that a trajectory will follow some vector lying in the convex hull $\mathcal{F}$ of the values, $\mathbf{f}_1$ and $\mathbf{f}_2$, that $\mathbf{f}$ takes either side of the discontinuity. That is, we resolve the jump at $\mathcal{D}$ by looking for trajectories that are solutions to the differential inclusion $\dot{\mathbf{x}} \in \mathcal{F}$. If $\mathcal{F}$ contains a vector lying tangent to the discontinuity manifold, then we say there exists a *sliding vector* $\mathbf{f}_\star$ that allows a trajectory to evolve along $\mathcal{D}$. It is readily observed that sliding occurs when the vector field either points towards the discontinuity manifold (so it is attractive, figure 1(ii)) or away from it (so it is repulsive, figure 1(iii)) on both sides.
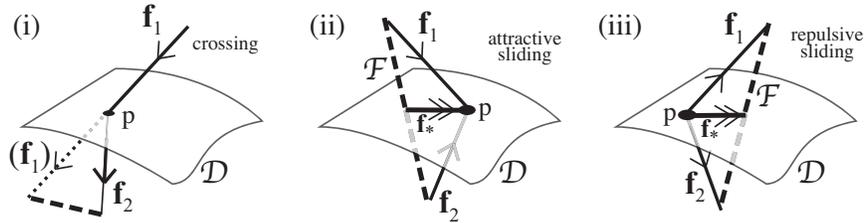

Figure 1: Dynamics at a discontinuity. A vector field switches between $\mathbf{f}_1$ and $\mathbf{f}_2$ either side of a manifold $\mathcal{D}$. By assuming that the vector field at the discontinuity is a linear combination $\mathcal{F}$ of vectors $\mathbf{f}^+$ and $\mathbf{f}^-$ (dotted), we find that the flow can cross through (i), or slide along (ii-iii) the manifold; the sliding vector is $\mathbf{f}_\star$ (double arrows).

If a discontinuity takes place on a surface that is an intersection of manifolds, then Filippov's approach does not generally lead to unique dynamical solutions. Consider the scenario in figure 2, where the point p lies at the intersection of two discontinuity manifolds, $\mathcal{D}_1$ and $\mathcal{D}_2$. The vector field switches between four different values $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \mathbf{f}_4$, at p. The convex set $\mathcal{F}$ obtained by drawing each vector from the point p, and stretching a hull over their endpoints, is three dimensional (shaded in figure 2), and so there can exist a one-parameter family (dashed) of vectors in $\mathcal{F}$ that lie tangent to the intersection $\mathcal{D}_1 \cap \mathcal{D}_2$ (which is a line or curve). This provides an infinite number of vectors that trajectories of $\dot{\mathbf{x}} \in \mathcal{F}$ can follow along the intersection.

The situation would be resolved if the three dimensional convex hull $\mathcal{F}$ could be reduced to a lower dimensional convex 'canopy' $\mathcal{F}$, as represented
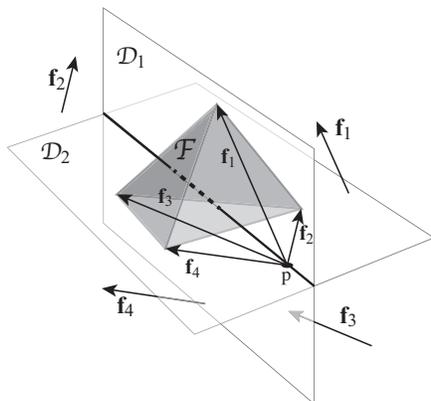
Figure 2: Convex combination of vector fields $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \mathbf{f}_4$, separated by discontinuity manifolds $\mathcal{D}_1$ and $\mathcal{D}_2$ which comprise $\mathcal{D}$. At the intersection we consider the convex hull of the four vector fields, a volume $\mathcal{F}$ (shaded). In this example $\mathcal{F}$ overlaps the intersection $\mathcal{D}_1 \cap \mathcal{D}_2$ along a line segment (dashed).

in figure 3. The intersection (a line) pierces the canopy (a surface) at a single point, which represents an isolated sliding vector. The purpose of this paper is to show that a natural choice for this reduced set of vectors typically exists, in arbitrary dimensions and for any number of (generically) intersections manifolds.
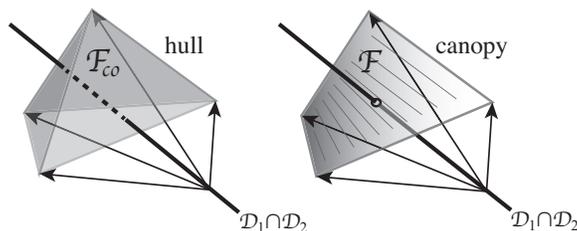

Figure 3: A convex combination of four vectors forms a tetrahedron $\mathcal{F}$ (left). Taking convex combinations in a pairwise manner instead forms a canopy $\mathcal{F}$ (right). The bold line represents the intersection of two manifolds $\mathcal{D}_1 \cap \mathcal{D}_2$ from figure 2, which intersects the hull along a set of points (dashed) and the canopy at a unique point (circled).

The classic texts [11, 24] discuss cases in which the necessary reduction in the dimension of the hull $\mathcal{F}$ is achieved because the system takes a special form, for example the typical control form $\mathbf{f} = \mathbf{a} + B\mathbf{u}$ where $\mathbf{u}$ is a vector of switches, which we visit at the end of section 2. Our purpose here is to show extend such reduction beyond special cases. The method may suggest new strategies for control if canopy-derived solutions are found to be sufficiently robust; indeed, our preliminary simulations in section 7 reveal no difficulty

in simulating them. Such generalizations are of interest beyond control, however, for example in applications to switching behaviour in biology and mechanics (see e.g. [7, 18, 20]).

In [8, 9], Filippov's method is refined by reducing the dimension of $\mathcal{F}$ as suggested above. The method is motivated by certain attractivity assumptions and restrictions on the number of intersecting manifolds, but the approach can actually be applied far more generally, extending straightforwardly to the intersection of m manifolds in at least m dimensions for any finite (positive) integer m. The more general approach opens a new world of *dummy dynamics* inside the intersection, and leads to *multiple* or *nonlinear sliding modes*, which will be introduced here.

The dummy system is one of the most novel parts of this paper. The method described here permits the flow at an intersection to take multiple, but finitely many, different forms. In such cases, a dummy system determines which trajectory the system will evolve along from given a set of initial conditions. This takes place on a dummy time-scale inside the switching surface, and occurs instantaneously in the perspective of the full system. The dummy dynamics can be studied using traditional dynamical systems methods, and thus any multiplicity of the flow at a switching intersection is resolved. The dummy time-scale can be interpreted as the limit of a fast time-scale (as studied in [16] but not pursued here), which both provides a motivation for the dummy dynamics, and allows a more in-depth analysis using singular perturbation techniques. It remains a future challenge to understand the role of such dynamics in applications. As we remark in the concluding section, non-uniqueness is an inescapable and important property of discontinuous systems. One current problem is to understand how such non-uniqueness can be interpreted dynamically, and, at a switching intersection, multiple sliding modes and dummy dynamics appear to provide this as far as possible.

Thus we derive a general framework for the most natural way to extend a vector field across an intersection of discontinuities, requiring no restrictions of attractivity. We also present the corresponding result in the framework of Utkin's equivalent control [23], where the discontinuity occurs not in $\mathbf{f}$ itself but in some parameter $\mathbf{g}$, so that $\mathbf{f}(\mathbf{x}, \mathbf{g})$ is continuous in $\mathbf{g}$ but the argument $\mathbf{g}$ itself is discontinuous. In doing so we review the two key approaches to discontinuities in flows [11, 23], making a useful distinction between *field combinations* (concerning a discontinuous vector field $\mathbf{f}$) and *parametric combinations* (concerning a discontinuous parameter $\mathbf{g}$ in the argument of $\mathbf{f}$).

Alternative ways of prescribing dynamics at a discontinuity can be found

in [3, 9, 11, 13, 14, 24], but the prevalent method from Filippov's work, which results in deterministic dynamics in wide classes of systems and proves broadly applicable, now forms the basis of the modern theory of this branch of *piecewise-smooth dynamical systems* [11, 7, 18, 25], particularly in taking it beyond control design into the study of various biological and physical systems (see e.g. [7, 18, 25]). These so-called *Filippov systems* will therefore be our sole concern here, i.e. continuous time systems where $\mathbf{f}$ suffers jumps as $\mathbf{x}$ varies. This excludes, for example, so-called *hybrid* systems (which include maps applied to $\mathbf{x}$) or *piecewise continuous* systems (where $\mathbf{f}$ is non-differentiable but still continuous).

We formulate the problem in section 2, and the simple route to its solution in sections 3-4, which is the main result of this paper. In section 5 we interpret the result in terms of hierarchies of switches, and we extend it to the equivalent control method in section 6. We illustrate the result with examples and novel attractors in section 7, and indicate exciting areas for further study in section 8.

2. The multiple switching problem

Filippov set out the means by which a system of differential equations can be made solvable at a discontinuity [11], paraphrased below in a manner suitable to our study. Consider a system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) = \{\ \mathbf{f}_i(\mathbf{x})\ \text{ when } \ \mathbf{x} \in R_i,\ \mathbf{x} \notin \mathcal{D}\ \}_{i=1,2,..,n} \tag{1}$$

where each $\mathbf{f}_i \in \mathbb{R}^N$ is a vector field depending smoothly on $\mathbf{x} \in \mathbb{R}^N$, and where the dot denotes differentiation with respect to a time t. The vector field $\mathbf{f}$ switches between the different functional forms $\mathbf{f}_i$ at the discontinuity surface $\mathcal{D}$, which is either a manifold or the union of finitely many, possibly intersecting, manifolds, which divides $\mathbb{R}^N$ into open regions $R_1, R_2, ..., R_n$. For simplicity we assume throughout this paper that the vector fields $\mathbf{f}_i$ pierce $\mathcal{D}$ transversally. There are two different problems concerning the well-posedness of the system (1), firstly whether it can be extended to $\mathbf{x} \in \mathcal{D}$ in a unique way, and secondly whether the flow of that extended system is unique. Our aim here is to define an extended system that does have a unique flow, at least in a wide class of systems (excepting certain singularities).

Let us assume there exists a set $\mathcal{F}$ that contains all physically reasonable values of $\mathbf{f}$ at a point on such an intersection. How we define this set is considered at the end of this section and the start of the next. Before

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confronting the defintion of $\mathcal{F}$, to extend the dynamical system (1) to the discontinuity set $\mathcal{D}$ we now define a differential inclusion

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \in \mathcal{F}(\mathbf{x}) \ , \tag{2}$$

which with (1) implies that $\mathcal{F}$ reduces to $\mathbf{f}_i$ in each region R_i. Our aim is then to find a definition of $\mathcal{F}$ on the switching threshold $\mathcal{D}$ that provides unique dynamics there. We start by defining solution trajectories via the set-valued system (2) (following [4, 11]).

Definition 1. *An $\mathcal{F}$-trajectory of (1) is a continuous piecewise-differentiable t-parameterized curve $\mathbf{x}(t)$ that satisfies (2) almost everywhere.*

Usually we will just write 'trajectory', assuming that a set $\mathcal{F}$ has been specified. We then need a more explicit way of expressing $\mathcal{D}$. In N dimensions, let the discontinuity surface $\mathcal{D}$ be the union of finitely many manifolds $\mathcal{D}_j$, each given by

$$\mathcal{D}_j = \left\{ \mathbf{x} \in \mathbb{R}^N : \ h_j(\mathbf{x}) = 0 \right\} \tag{3}$$

for smooth scalar functions h_j, where the gradient vectors $\nabla h_1, \nabla h_2, ...$ are all linearly independent. If m manifolds of dimension $N - 1$ intersect transversally at some point then we must have $m \leq N$. In the neighbourhood of such a point the function $\mathbf{f}$ switches across $\mathcal{D}$ between 2^m different values $\mathbf{f}_1, \mathbf{f}_2, ..., \mathbf{f}_{2^m}$.

A trajectory that crosses $\mathcal{D}$ transversally will switch instantaneously between the vector fields $\mathbf{f}_i$, and satisfies definition 1 without the specific form of $\mathcal{F}$ needing to be specified. The only alternative is that a trajectory slides along $\mathcal{D}$, and then we need to define the vector field it will follow, known as the *sliding vector field*. The standard definition (see e.g. [11]) can be written as follows.

Definition 2. *A sliding vector is some $\mathbf{f}_\star \in \mathcal{F}$ that lies tangent to $\mathcal{D}$. In particular, at a point $\mathbf{x}$ on the intersection of m different manifolds $\mathcal{D}_1, \mathcal{D}_2, ... \mathcal{D}_m$, a sliding vector $\mathbf{f}_\star$ satisfies*

$$\mathbf{f}_\star(\mathbf{x}) \in \mathcal{F}(\mathbf{x}) \quad \text{such that} \quad \mathbf{f}_\star(\mathbf{x}) \cdot \nabla h_j(\mathbf{x}) = 0 \quad \forall j = 1, 2, ..., m \ . \tag{4}$$

To go any further, it is now necessary to complete our definition of the set $\mathcal{F}$. To conclude this section we state the standard choice for $\mathcal{F}$, which we label $\mathcal{F}_{co}$, before refining this choice in the following section.

Definition 3. *In the neighbourhood of a point* $\mathbf{x} \in \mathcal{D}$ *where* $\mathbf{f}$ *switches between values* $\{\mathbf{f}_i(\mathbf{x})\}_{i=1,..,2^m}$*, let* $\mathcal{F}_{co}$ *be the* convex hull *of* $\mathbf{f}$*, given by*

$$\mathcal{F}_{co}(\mathbf{x}) \;=\; \{\, \mathbf{f} = \sum_{i=1}^{2^m} \lambda_i \mathbf{f}_i(\mathbf{x}) : \;\; \lambda_i \in [0,1] \,\} \,, \tag{5}$$

where $\lambda_i = 1$ *for* $\mathbf{x} \in \mathcal{R}_i$*, and the coefficients* λ_i *are subject to a normalization condition* $\lambda_1 + \lambda_2 + ... + \lambda_{2^m} = 1$*.*

The convex hull $\mathcal{F}_{co}$ then subsumes the original righthand-side in (1), and in addition provides a reasonable set of velocity vectors that a trajectory may follow at $\mathcal{D}$. To resolve the discontinuity we assume that $\mathbf{f}$ takes some value lying in $\mathcal{F}_{co}$, and we attempt to fix the values of the λ_i's using definitions 1-2 (for details of the relevant theory of differential inclusions see [11]). However, we have immediately:

Lemma 1. *At a point where* $\mathcal{D}$ *consists of* $m > 1$ *intersecting manifolds, in the generic situation when the matrices* $\{\mathbf{f}_i \cdot \nabla h_j\}$ *and* $\{\mathbf{f}_i\}$ *have full rank with* $i = 1, .., 2^m$*, and* $j = 1, .., m$*, the system* $\dot{\mathbf{x}} \in \mathcal{F}_{co}$ *has either no sliding vectors or infinitely many.*

Proof. Substituting $\mathcal{F} = \mathcal{F}_{co}$ into definition 2, since $\partial(\mathbf{f} \cdot \nabla h_j)/\partial \lambda_i = \mathbf{f}_i \cdot \nabla h_j$ and the matrix $\{\mathbf{f}_i \cdot \nabla h_j\}$ has full rank, the implicit function theorem implies the conditions (4) fix m of the unknowns λ_i, leaving a further $2^m - 1 - m$ unknown. If the matrix $\{\mathbf{f}_i\}$ also has full rank then the vectors $\mathbf{f}_i$ are linearly independent, so $\mathbf{f}$ spans a set of dimension $\min(n, 2^m{-}1{-}m)$ as the remaining unknown λ_i's vary over the intervals $[0,1]$. Then for $m > 1$ there are no solutions if the λ_i's do not all lie in $[0,1]$ for any $\mathbf{f}$, or there are inifinitely many solutions if the λ_i's all lie in $[0,1]$ for any $\mathbf{f}$. For $m = 1$ the λ_i's are uniquely determined; this is the simplest case where $\mathcal{D} = \mathcal{D}_1$ is a smooth manifold as in figure 1. □

Figure 2 in section 1 illustrates this for $m = 2$. In the next section we argue that a more specific set than (5) is actually more natural, and that it reduces the dimension of the convex hull to an m-dimensional set we call the *convex canopy*, from which isolated sliding trajectories can be found.

Some special cases are mentioned in [11] where breaking the genericity conditions in Lemma 1 leads to finitely many sliding solutions; note that these may still lead to multiple sliding solutions, lacking the criterion to choose between them which we introduce in section 4. One case is common in linear control, when $\mathbf{f} = \mathbf{a} + B\mathbf{u}$ or equivalently $\mathbf{f} = \mathbf{a}_{u_1} + \mathbf{a}_{u_2} + ... + \mathbf{a}_{u_m}$,

where $\mathbf{a}_j \in \mathbb{R}^N$, $\mathbf{u} = (u_1, .., u_m)^\mathsf{T} \in \mathbb{R}^m$, $B \in \mathbb{R}^{N \times m}$, and each u_j switches between 0 and 1 at its associated manifold $h_j = 0$; then there are only m unknowns u_j and they are well-determined, since from $\{\mathbf{f}_i \cdot \nabla h_j\}$ we can extract an invertible $m \times m$ matrix whose rows determine the remaining $2^m - 1 - m$ rows, see [11]. A less obvious case occurs if $\{\mathbf{f}_i \cdot \nabla h_j\}$ consists of k linearly independent vectors with $k > m$, but the constraints $\lambda_i \in [0, 1]$ can only be satisfied if $2^m - 1 - k$ of the λ_i's are equal to 0 or 1, then (4) fixes the remaining k of the λ_i's and the system is fully determined; e.g. if $m = 2$ and defining $\mathbf{f}_i^\perp = (\mathbf{f}_i \cdot \nabla h_1, \mathbf{f}_i \cdot \nabla h_2)$ with $\mathbf{f}_1^\perp = -r\mathbf{f}_2^\perp$ for some $r > 0$, then solving the sliding problem at the intersection $h_1 = h_2 = 0$ gives $0 = (\lambda_1 - r\lambda_2)\mathbf{f}_1^\perp + \lambda_3\mathbf{f}_3^\perp + \lambda_4\mathbf{f}_4^\perp$, solveable to find $\lambda_3 = \frac{\mathbf{f}_1^\perp \times \mathbf{f}_4^\perp}{\mathbf{f}_3^\perp \times \mathbf{f}_1^\perp}\lambda_4$, then if the fraction is positive $\lambda_i \in [0, 1]$ can only be satisfied for $\lambda_3 = \lambda_4 = 0$ and the sliding problem has a definite solution, otherwise the fraction is negative and, as in the Lemma, there exist a continuum of solutions.

3. The convex canopy

By labelling the functions $\mathbf{f}_i$ a little differently, we can express the 2^m independent values of $\mathbf{f}$ in (1) as a system of m independent switches. Replace the sequence of 2^m values $\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \mathbf{f}_4, ...$, with a binary representation $\mathbf{f}_{000..}, \mathbf{f}_{100..}, \mathbf{f}_{010...}, \mathbf{f}_{110...}, ...$, where the jth index is assigned so that it flips between 0 and 1 across the manifold labelled $\mathcal{D}_j$. Only the number base of the index for $\mathbf{f}_j$ has changed here, so no generality is lost. The function $\mathbf{f}$ can then be written as a sum:

Proposition 2. *The piecewise-defined function* $\mathbf{f}$ *in (1) can be written as*

$$\mathbf{f}(\mathbf{x}; \mu_1, \mu_2, .., \mu_m) = \sum_{p_1=0}^{1} \sum_{p_2=0}^{1} ... \sum_{p_m=0}^{1} \mu_1^{(p_1)} \mu_2^{(p_2)} ... \mu_m^{(p_m)} \mathbf{f}_{p_1 p_2 ... p_m}(\mathbf{x}) , \qquad (6)$$

where each μ_j *can take values 0 or 1 only, and we introduce a shorthand*

$$\mu_j^{(0)} = \mu_j , \qquad \mu_j^{(1)} = 1 - \mu_j .$$

Proof. Each of the 2^m different values $\mathbf{f}(\mathbf{x}; \mu_1, \mu_2, ...)$ maps one-to-one to the original functions $\mathbf{f}_{\mu_1 \mu_2 ..}$, as is verified directly by expanding out the series (6). Note that since the μ_j's appear in pairs μ_j and $1 - \mu_j$ whose sum is unity, the coefficients of all $\mathbf{f}_i$'s automatically sum to unity. $\square$

With the function $\mathbf{f}$ expressed in the form (6) (which is an extension of the form for $m = 2$ given in [9]), there is an obvious way to derive a convex

set from the values $\mathbf{f}_i$. The switch that takes place at each manifold $\mathcal{D}_j$ flips the jth index in $\mathbf{f}_{\mu_1\mu_2..\mu_j..}$, and switches the parameter μ_j between 0 and 1. A convex set can therefore be defined for each switch, by replacing each μ_j with a parameter λ_j that varies continuously over $\lambda_j \in [0, 1]$.

Definition 4. *In the neighbourhood of a point* $\mathbf{x} \in \mathcal{D}$ *where* $\mathbf{f}$ *switches between values* $\{\mathbf{f}_i(\mathbf{x})\}_{i=1,..,2^m}$, *let* $\mathcal{F}$ *be the* convex canopy *of* $\mathbf{f}$, *defined as*

$$\mathcal{F}(\mathbf{x}) = \{\ \mathbf{f}(\mathbf{x}; \lambda_1, \lambda_2, .., \lambda_m) : \ \lambda_j \in [0, 1], \ j = 1, 2, ..., m\ \}\ , \qquad (7)$$

in terms of the function $\mathbf{f}(\mathbf{x}; \lambda_1, \lambda_2, ..)$ *as defined in (6).*

The use of this canopy to find the dynamics of the discontinuous system constitutes the central result of this paper, but to put it into practice will require results to come in section 4. The principles can be summarized as follows. In place of the system (1) we consider

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})\ , \qquad (8)$$

where

$$\mathbf{f}(\mathbf{x}) \in \mathcal{F}(\mathbf{x}) = \left\{ \mathbf{f}(\mathbf{x}; \lambda_1, \lambda_2, .., \lambda_m) : \ \begin{array}{ll} \lambda_j \in [0, 1] & \text{for} \quad \mathbf{x} \in \mathcal{D} \\ \lambda_j = 0 \text{ or } 1 & \text{for} \quad \mathbf{x} \notin \mathcal{D} \end{array} \right\}\ , \qquad (9)$$

in terms of the function $\mathbf{f}(\mathbf{x}; \lambda_1, \lambda_2, ..)$ defined in (6). The righthand side of (8) is smooth and single-valued in a neighbourhood of any point $\mathbf{x} \notin \mathcal{D}$. At any point $\mathbf{x} \in \mathcal{D}$ at which m manifolds $\mathcal{D}_j$ intersect, the righthand side of (8) generically lies in an m dimensional convex canopy.

We are particularly interested in whether there exist sliding trajectories, for which we directly apply definition 2 using (7), thus the velocity vectors of sliding are found by solving the equations $\mathbf{f} \cdot \nabla h_j = 0$ for every $j = 1, 2, ..., m,$, subject to the restriction $\lambda_j \in [0, 1]$. Because this is a system of m equations in m unknowns λ_j, if sliding vectors $\mathbf{f}_* \in \mathcal{F}$ exist then they are typically isolated and therefore well-defined. However, we have the following result.

Proposition 3. *At a point where* $\mathcal{D}$ *consists of* $m > 1$ *intersecting manifolds and* $\{\partial \mathbf{f} \cdot \nabla h_j / \partial \lambda_i\}_{i,j=1,..,m}$ *has full rank,* $\dot{\mathbf{x}} \in \mathcal{F}$ *has up to* $m!$ *well-defined sliding vectors.*

Proof. With $\mathcal{F}$ defined by (6) and (7), since $\{\mathbf{f}_i \cdot \nabla h_j\}$ has full rank the m equations in (4) are non-degenerate, and thus consist of an algebraic system of m simultaneous equations, linear in each of m different λ_i's (but containing products of different λ_i's). If we solve the 1st equation for λ_1, we

find there is at most one real solution for λ_1 in terms of the remaining $m-1$ λ_i's, then two solutions for λ_2 in terms of the remaining $m-2$ λ_i's, then three solutions for λ_3 in terms of the remaining $m-3$ λ_i's, and so on. The combination of these gives at most $1 \times 2 \times 3 \times ...m = m!$ different real-valued solutions for the complete set $\{\lambda_1, \lambda_2, ..., \lambda_m\}$. $\qquad\square$

In the following sections we discuss the multiplicity of these sliding vectors, and see how the dynamics distinguishes between them. The form of (6) will also be interpreted in terms of hierarchies of sliding vector fields, followed by a further generalization of the canopy method where switching occurs via non-linear control variables, and finally we shall present some illustrative examples. Let us first end this section with an example that demonstrates the canopy and exhibits multiple sliding vectors.

Example 1. Consider the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ in coordinates $\mathbf{x} = (X, Y, Z)$. Let $\mathbf{f}$ switch between values

$$
\begin{aligned}
\mathbf{f}_{00} &= (-1, 4\alpha/5, \zeta), & \mathbf{f}_{10} &= (1/2, -4/5, \zeta), \\
\mathbf{f}_{11} &= (1, 4\alpha/5, \xi), & \mathbf{f}_{01} &= (-1/2, -4/5, \xi) ,
\end{aligned}
\tag{10}
$$

across two switching manifolds $\mathcal{D}_1$ and $\mathcal{D}_2$ given by the coordinate planes $X = 0$ and $Y = 0$ respectively. The quantities ζ and ξ are arbitrary (constants or smooth functions), while α is a constant. Each vector field $\mathbf{f}_{p_1 p_2}$ applies in a region labelled $R_{p_1 p_2}$, and the jth index p_j switches between 0 and 1 across the manifold $\mathcal{D}_j$. We can denote the sliding vector fields on $\mathcal{D}_1$ and $\mathcal{D}_2$ (if they exist by definition 2) using the notation:

$$
\mathbf{f} = \begin{cases} \mathbf{f}_{\star 0} \text{ for } \mathbf{x} \in \{\mathcal{D}_1 : Y > 0\} & \mathbf{f}_{\star 1} \text{ for } \mathbf{x} \in \{\mathcal{D}_1 : Y < 0\} \\ \mathbf{f}_{0\star} \text{ for } \mathbf{x} \in \{\mathcal{D}_2 : X > 0\} & \mathbf{f}_{1\star} \text{ for } \mathbf{x} \in \{\mathcal{D}_2 : X < 0\} \end{cases} .
$$

We delay the calculation of these to the next section.

We will denote the sliding vector field on the intersection of $\mathcal{D}_1$ and $\mathcal{D}_2$ as $\mathbf{f}_{\star\star}$. To find sliding vectors $\mathbf{f}_{\star\star}$ on the intersection, we construct the canopy combination (from (6))

$$
\mathbf{f}_{\star\star} = \lambda_2 [\lambda_1 \mathbf{f}_{00} + (1 - \lambda_1) \mathbf{f}_{10}] + (1 - \lambda_2)[\lambda_1 \mathbf{f}_{01} + (1 - \lambda_1) \mathbf{f}_{11}] .
$$

The canopy at a point $p \in \mathcal{D}_1 \cap \mathcal{D}_2$ is shown in figure 4 for $\alpha = 5/4$ (left) and $\alpha = 5/8$ (right). We then seek solutions for $\lambda_1, \lambda_2 \in [0, 1]$ such that $\mathbf{f}_{\star\star}$ lies in the tangent space of the intersection, meaning (from (4)) that

$$
\mathbf{f}_{\star\star} \cdot \nabla X = 0 \quad \text{and} \quad \mathbf{f}_{\star\star} \cdot \nabla Y = 0 .
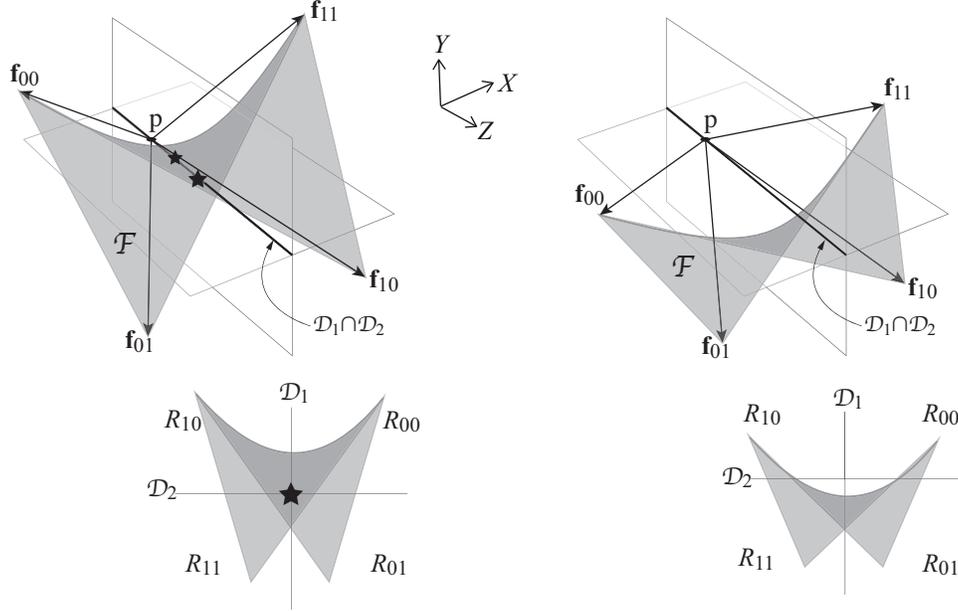$$

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Figure 4: The bifurcation of sliding vectors from a canopy in three dimensions. The canopy construction joins each pair of vertices whose vectors $\mathbf{f}_{ij}$ share a common index i or j, which in general results in a curved surface (shaded). Left: the intersection of the switching surfaces $\mathcal{D}_1 \cap \mathcal{D}_2$ pierces the canopy $\mathcal{F}$ at two points (starred). Right: the intersection misses the canopy. This is easier to see in the end-on views below the main figures, where the regions R_{ij} and switching surfaces $\mathcal{D}_k$ are also labelled.

Solving the simultaneous equations gives

$$\lambda_1 = \frac{1}{2}\left(1 \mp \sqrt{\frac{\alpha - 1}{3(\alpha + 1)}}\right) , \qquad \lambda_2 = \frac{1}{2}\left(1 \pm \sqrt{\frac{3(\alpha - 1)}{\alpha + 1}}\right) ,$$

(note these are independent of ζ and ξ), hence there exist a pair of solutions for $|\alpha| > 1$ and no solutions (i.e. no sliding vector field) for $|\alpha| < 1$. In the former case, the vector $\mathbf{f}_{\star\star}$ gives sliding along the intersection with two different possible speeds $\mathbf{f}_{\star\star} \cdot \nabla Z = \xi + \frac{1}{2}(\zeta - \xi)\left(1 \pm \sqrt{3(\alpha - 1)/(\alpha + 1)}\right)$.

4. The multiple sliding problem: attractivity & dummy dynamics

What we have termed the 'convex' canopy (6), while being formed from convex combinations of the switches, is not a convex set in the sense discussed by Filippov, since when embedded in the coordinate space of $\mathbf{x}$, the

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line between two points in $\mathcal{F}$ does not lie entirely in $\mathcal{F}$. This has two important consequences, first for the existence of sliding vectors which we discuss below, and second for the existence of limit sets of the flow, which we comment on in a footnote[1].

Firstly, the canopy is generally a curved surface, so under generic conditions the equations (4) may have isolated solutions giving well-defined sliding vectors, and while typically finite in number, more than one sliding vector is possible (as in Example 1 above).

There is no simple criterion for determining a priori how many sliding vectors will exist in general (at least a general criterion is not yet known). One must solve the system and investigate how many valid vectors there are within the convex canopy $\mathcal{F}$ that are tangent to the discontinuity surface $\mathcal{D}$. In the special case when all of the local vector fields are directed towards the intersection (i.e. it is attractive), then unique sliding vectors do exist, and this case is considered in [8, 9]. In slightly weaker conditions, when the intersection attracts nearby dynamics and is reachable through sliding motion outside the intersection, uniqueness is possible as shown in [8].

In general, however, Proposition 3 raises the question: if there are up to $m!$ sliding vectors, possibly coexisting with up to 2^m crossing vectors, which vector will the system follow from a given initial condition? To answer this we construct a dummy system in the space of $\lambda_1, \lambda_2, ..., \lambda_m$.

Definition 5. *At a point* $\mathbf{x} \in \mathcal{D}_1 \cap \mathcal{D}_2 \cap ... \mathcal{D}_m$, *consider a dynamical system*

$$
\begin{aligned}
\lambda_1' &= \mathbf{f}(\mathbf{x}) \cdot \nabla h_1(\mathbf{x}) \\
\lambda_2' &= \mathbf{f}(\mathbf{x}) \cdot \nabla h_2(\mathbf{x}) \\
&\vdots \\
\lambda_m' &= \mathbf{f}(\mathbf{x}) \cdot \nabla h_m(\mathbf{x})
\end{aligned}
\tag{11}
$$

where the prime denotes differentiation with respect to a dummy time τ. Changes in finite τ-time are instantaneous in the t-timescale of (8).

---

[1]The more subtle consequence of the fact that $\mathcal{F}$ is not strictly convex regards the existence and regularity of trajectories. It is known (see [11], p.79) that without such convexity, the limit $\mathbf{x}_\infty(t)$ of a convergent sequence of trajectories $\{\mathbf{x}_r(t)\}_{r \in \mathbb{Z}}$ to the problem $\dot{\mathbf{x}}_r(t) \in \mathcal{F}(\mathbf{x}_r(t))$, need not itself be a trajectory of the system, $\dot{\mathbf{x}}_\infty(t) \notin \mathcal{F}(\mathbf{x}_r(t))$. It remains to be shown whether this irregularity of limits can be handled using the property that $\mathcal{F}$ *is* convex in the space of λ_j in the sense that any two vectors $\mathbf{f}_p, \mathbf{f}_q \in \mathcal{F}$, in a canopy $\mathcal{F}$ of dimension m, can be joined by m straight line segments, all of which lie entirely in $\mathcal{F}$, and which correspond to adjusting each coefficient λ_j, $j = 1, 2, ..., m$ in turn when $\mathbf{f}_p$ and $\mathbf{f}_q$ are written in the form (6). A rigorous investigation of this is beyond our scope here, and may deserve future study.

The righthand side of (11) lies in the tangent space of the intersection $\mathbf{x} \in \mathcal{D}_1 \cap \mathcal{D}_2 \cap ... \mathcal{D}_m \subset \mathcal{D}$ by (3). Definition 5 is an extension of the dummy system introduced at a switching manifold (without intersections) in [16].

Conjecture 4. *Take an initial condition*

$$(\mathbf{x}; \lambda_1, \lambda_2, ..., \lambda_m) = (\mathbf{x}_0; \Lambda_1, \Lambda_2, ..., \Lambda_m)$$

for which $h_j(\mathbf{x}_0) = 0$ and $\Lambda_j \in [0, 1]$ for all $j = 1, 2, ..., m$. Then a trajectory of (2), where $\mathcal{F}$ is the convex canopy (7), will evolve according to (11) until it reaches an attracting fixed point or limit set of (11), or reaches the threshold given by $\lambda_j = 0$ or 1 for any $j = 1, 2, ..., m$.

We refrain from proposing this conjecture as a strict theorem, because, as discussed in section 8, this important result can be motivated by many considerations, including discrete, perturbative, or stochastic approaches. In fact, the justification of even the basic Filippov method by such approaches remains open (see e.g. [16]), so a full understanding will require further study from various perspectives, too lengthy to embark on here. Instead we provide the following example, which gives distinct evidence in support of the conjecture via numerical simulation of a test system.

Example 2. Let us revisit the system from Example 1. Sliding vectors exist over the entire discontinuity surface $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$. Let us first find the attractivity of the sliding vector field on, say, the sliding region $\{\mathcal{D}_1 : y > 0\}$ that separates R_{00} from R_{10}. Let's denote this sliding vector by $\mathbf{f}_{\star 0}$, which by definition 2 must satisfy

$$\mathbf{f}_{\star 0} = \lambda_1 \mathbf{f}_{00} + (1 - \lambda_1) \mathbf{f}_{10} ,$$

the righthand of which is a member of the one-dimensional canopy (a line) with $\lambda_1 \in [0, 1]$. The dummy system on this surface is given from definition 5 by

$$\lambda_1' = \mathbf{f}_{\star 0} \cdot \nabla h_1$$

There is a sliding vector where this has a fixed point, i.e. where $\mathbf{f}_{\star 0} \cdot \nabla h_1 = 0$. The solution is $\lambda_1 = 1/3$, giving $\mathbf{f}_{\star 0} = (0, \frac{12}{5}(\alpha - 2), \zeta)$. This flows towards the intersection if $\alpha < 2$ and away from it if $\alpha > 2$. The eigenvalue of the fixed point is $\frac{d}{d\lambda_1} \mathbf{f}_{\star 0} \cdot \nabla h_1|_{\lambda_1 = 1/3} = -3/2$, hence the fixed point is attractive. So if a trajectory of (2) arrives at the sliding region $\{\mathcal{D}_1 : y > 0\}$, either from R_{10} in which $\lambda_1 = 0$ or from R_{00} in which $\lambda_1 = 1$, the value of λ_1 will jump to $\lambda_1 = 1/3$ via the dummy system, then the trajectory will continue to evolve along the sliding vector $\mathbf{f}_{\star 0}$.

A similar argument gives the attractivity on the other sliding regions: an attracting sliding vector field exists everywhere on $\mathcal{D}_1$ and $\{\mathcal{D}_2 : x < 0\}$, and a repelling sliding vector field exist on $\{\mathcal{D}_2 : x > 0\}$.

Let us now find the attractivity of sliding vector fields on the intersection, which for $\lambda_{1,2} \in [0,1]$ must satisfy

$$\mathbf{f}_{\star\star} = \lambda_2 \left\{\lambda_1 \mathbf{f}_{00} + (1 - \lambda_1)\mathbf{f}_{10}\right\} + (1 - \lambda_2)\left\{\lambda_1 \mathbf{f}_{01} + (1 - \lambda_1)\mathbf{f}_{11}\right\} ,$$

where the righthand side is the convex canopy. The dummy system for $\lambda_{1,2}$ is

$$\begin{aligned}
\lambda_1' &= \mathbf{f}_{\star\star} \cdot \nabla h_1 \\
\lambda_2' &= \mathbf{f}_{\star\star} \cdot \nabla h_2
\end{aligned}$$

which has a pair of fixed points at

$$(\lambda_1, \lambda_2) = \frac{1}{2}(1,1) \pm \frac{1}{2}\left(-1/\sqrt{3}, \ \sqrt{3}\right)\sqrt{\frac{\alpha - 1}{\alpha + 1}} ,$$

implying two coexisting sliding vector fields. Which will the dynamics follow? Let us fix $\alpha = 5/4$, then these become

$$(\lambda_1, \lambda_2) \approx (0.4, 0.8) \quad \text{and} \quad (\lambda_1, \lambda_2) \approx (0.6, 0.2) .$$

The Jacobian matrix of the dummy system at the fixed points is given by

$$\begin{pmatrix} \frac{d}{d\lambda_1}\mathbf{f}_{\star\star} \cdot \nabla h_1 & \frac{d}{d\lambda_2}\mathbf{f}_{\star\star} \cdot \nabla h_1 \\ \frac{d}{d\lambda_1}\mathbf{f}_{\star\star} \cdot \nabla h_2 & \frac{d}{d\lambda_2}\mathbf{f}_{\star\star} \cdot \nabla h_2 \end{pmatrix} = \begin{pmatrix} -\frac{3}{2} & -\frac{1}{2} \\ \pm\frac{3\sqrt{3}}{5} & \mp\frac{\sqrt{3}}{5} \end{pmatrix} ,$$

with eigenvalues $-0.9 \pm 0.4i$ for the fixed point at $(\lambda_1, \lambda_2) \approx (0.4, 0.8)$, which is therefore an attracting focus, and eigenvalues -0.6 ± 1.2 at $(\lambda_1, \lambda_2) \approx (0.6, 0.2)$, which is therefore a saddle. These are illustrated in figure 5. The
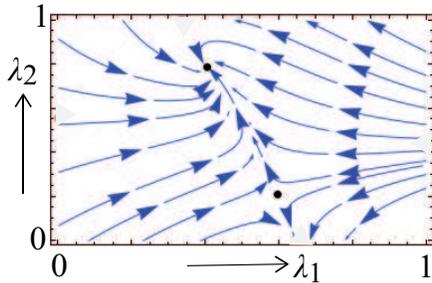

Figure 5: The flow of the dummy system (11) in the λ_1-λ_2 plane, on the square $[0,1] \times [0,1]$, contains an attracting focus and a saddle for $\alpha = 5/4$.

attracting focus at $(\lambda_1, \lambda_2) \approx (0.4, 0.8)$ is clearly the dominant attractor of the (λ_1, λ_2) dummy system. The sliding vector field for each state is then

$$\mathbf{f}_{\star\star}|_{(0.4,0.8)} = \left(0,\ 0,\ \frac{1}{\sqrt{3}}\right)\ , \qquad \mathbf{f}_{\star\star}|_{(0.6,0.2)} = \left(0,\ 0,\ -\frac{1}{\sqrt{3}}\right)\ .$$

So if a trajectory follows $\mathbf{f}_{\star\star}|_{(0.4,0.8)}$ as predicted by Conjecture 4, it will move along the intersection in the positive Z-direction. Simulations verify this, and an example is shown in figure 6.
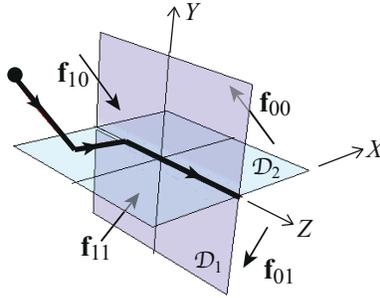

Figure 6: A simulation of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}; \lambda_1, \lambda_2, ..., \lambda_m)$ with $\alpha = 5/4$, $\zeta = 1$, $\xi = -1$. The orbit of an initial condition in R_{10} hits $\mathcal{D}_2$ and slides to the intersection, then slides along the intersection in the positive Z direction. The directions of the vector fields $\mathbf{f}_{ij}$ are indicated. The simulation is made by approximating each discontinuous λ_j by a sigmoid function $\lambda_j \mapsto [1 + \tanh(kh_j)]/2$ with $k = 10^6$, and solving the initial value problem numerically using Mathematica$^{\text{TM}}$ function NDSolve.

A much more compelling demonstration of Conjecture 4 can be obtained if we extend this system to 4 dimensions, so that the intersection $\mathcal{D}_1 \cap \mathcal{D}_2$ is two-dimensional and can exhibit more interesting dynamics. Let us take coordinates $\mathbf{x} = (X, Y, Z, W)$, and consider the system

$$\begin{aligned}
\mathbf{f}_{00} &= (-1, 4\alpha/5, W, -Z), & \mathbf{f}_{10} &= (1/2, -4/5, W, -Z), \\
\mathbf{f}_{11} &= (1, 4\alpha/5, -W, -Z), & \mathbf{f}_{01} &= (-1/2, -4/5, -W, -Z)\ ,
\end{aligned}$$

then all of the analysis of the dummy system above applies directly, except now the sliding vector fields on the intersection are given by

$$\mathbf{f}_{\star\star}|_{(0.4,0.8)} = \left(0,\ 0,\ \frac{W}{\sqrt{3}},\ -Z\right)\ , \quad \mathbf{f}_{\star\star}|_{(0.6,0.2)} = \left(0,\ 0,\ -\frac{W}{\sqrt{3}},\ -Z\right)\ ,$$

the former of which is a center in the (Z, W) plane, the latter is a saddle. Instead of just evolving along the Z-axis in a different direction, a trajectory that enters the intersection will evolve in the coordinates (Z, W) (with $X = Y = 0$ fixed inside the intersection). Conjecture 4 predicts that the

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trajectory will satisfy $(\lambda_1, \lambda_2) \approx (0.4, 0.8)$ corresponding to the attractor of the dummy system. In figure 7 we simulate a trajectory corresponding to that in figure 6, just plotting its (Z, W) coordinates. The left and right panels underlay this with the flows in the vector fields $\mathbf{f}_{\star\star}|_{(0.4,0.8)}$ and $\mathbf{f}_{\star\star}|_{(0.6,0.2)}$ respectively, confirming rather strikingly how the trajectory rotates around the center predicted inside the intersection, consistent with Conjecture 4.
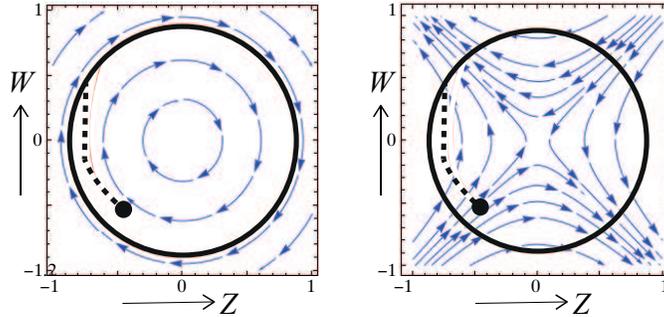

Figure 7: Dynamics on the Z-W coordinates at the intersection $X = Y = 0$. Left: the sliding vector field $\mathbf{f}_{\star\star}|_{(0.4,0.8)}$ associated with the attractive fixed point of the dummy system. Right: the sliding vector field $\mathbf{f}_{\star\star}|_{(0.6,0.2)}$ associated with the repulsive fixed point of the dummy system. A trajectory of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}; \lambda_1, \lambda_2, ..., \lambda_m)$ is shown, solved as in figure 6 for the 4 dimensional problem. The dotted curve shows the trajectory before it reaches the intersection, after which it is seen to rotate (bold curve) according to the sliding vector field on the left, ignoring the system on the right.

A remark is necessary on the simulations in figure 7. The trajectory shown is not obtained by instructing a numerical integrator to follow any of the sliding vector fields $\mathbf{f}_{\star\star}$ defined above. Rather, it is obtained by smoothing the discontinuous system (1) using the vector fields (10), approximating each discontinuous λ_j by a sigmoid function $\lambda_j \mapsto [1 + \tanh(kh_j)]/2$ with $k = 10^6$, and solving the initial value problem numerically using Mathematica$^{\mathrm{TM}}$ function NDSolve. This approximation of the ideal discontinuous problem follows the attractor predicted by the dummy system of the ideal problem. An alternative is to use event-driven methods, see e.g. [10, 19], but at present these have certain limitations, in particular in how they should handle singularities such as one we shall see in section 7.4.

5. Interpretation: Sliding hierarchy and switching isonomy

The result in (7) is actually just an intuitive application of Filippov's method for a 2-mode system, carried out iteratively to build a hierarchical 2^m-mode system.

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To see this, consider a problem where $\mathbf{f}$ switches between $\mathbf{f}_0$ and $\mathbf{f}_1$ across a discontinuity manifold $\mathcal{D}_1$. We can solve this using Filippov's usual method for a system with a discontinuity along a manifold (figure 1), where typically we can find a sliding vector field $\mathbf{f}_\star$ that exists on certain open regions of $\mathbf{x} \in \mathcal{D}_1$.

Now imagine we are subsequently informed that $\mathbf{f}_0$ and $\mathbf{f}_1$ are themselves nonsmooth, in fact $\mathbf{f}_0$ switches between values $\mathbf{f}_{00}$ and $\mathbf{f}_{01}$, while $\mathbf{f}_1$ switches between values $\mathbf{f}_{10}$ and $\mathbf{f}_{11}$; these new switches take place across a manifold $\mathcal{D}_2$. This means that the sliding vector field $\mathbf{f}_\star$ found in the previous paragraph must actually switch between two values, say $\mathbf{f}_{\star 0}$ and $\mathbf{f}_{\star 1}$, where $\mathcal{D}_2$ intersects $\mathcal{D}_1$.

We can iterate this process, adding manifolds $\mathcal{D}_j$ that introduce a jth index of $\mathbf{f}_{01\ldots}$, which flips between 0 and 1, finding that each of the sliding vector fields is also discontinuous at each new intersection. We continue until all m manifolds have been considered. Crucially, we only solve for the λ_j's at an intersection once the canopy combination (7) has been derived.

The sliding vector fields obtained in this way are typically well-defined, because (8) with (4) determines the unknowns λ_j exactly. What is not obvious is why the hierarchical description above should not depend on the order in which we consider the manifolds $\mathcal{D}_j$, however it is immediately clear that the resulting expression (6) treats each λ_j and each $\mathbf{f}_{p_1 p_2 \ldots}$ equally. Thus in the scheme described above we could have arranged the manifolds $\mathcal{D}_j$ in any order, combining the vector fields cumulatively (taking combinations of combinations of ... vector fields), and the final result contains no memory of any ordering.

The sliding vectors obtained in this way are not the same as would be obtained by finding the vector fields on each $\mathcal{D}_j$ separately, and combining these a posteriori to find vectors lying along the intersection; that method depends on the order in which one takes combinations across each of the manifolds $\mathcal{D}_j$. The reason that (6) is able to form an orderless hierarchy lies in the fact that each λ_j as a function of $\mathbf{x} \in \mathcal{D}_j$ will have a discontinuity at the intersections, and the values of λ_j at the intersections is determined only after all combinations have been taken to form the convex canopy. This ensures that each of the switches at each of the manifolds $\mathcal{D}_j$ is treated isonomously, and in such a situation, when (1) contains no suggestion that any $\mathbf{f}_i$ should be weighted more strongly in forming a combination of $\mathbf{f}_i$'s than any other, the convex canopy method seems to be not only physically appropriate, but the natural generalization of Filippov's method to multiple switching surfaces.

6. The equivalent control method

Instead of $\mathbf{f}$ explicitly undergoing a switch at $\mathcal{D}$, we could assume that $\mathbf{f}$ is a continuous function of some 'control' vector $\mathbf{g}$. We can then assume that $\mathbf{g}$ undergoes a switch at $\mathcal{D}$. The results above carry over in a similar form as follows.

As we did with $\mathcal{F}$, we can define a set $\mathcal{G}$ that is supposed to contain physically reasonable values of the input $\mathbf{g}$, by taking the convex hull of all values $\mathbf{g}_1, \mathbf{g}_2, ..\mathbf{g}_m$, which the function $\mathbf{g}$ switches between at $\mathcal{D}$. The convex hull is given by

$$\mathcal{G}_{co} = \{\mathbf{g} = \sum_{i=1}^{2^m} \lambda_i \mathbf{g}_i : \ \lambda_i \in [0,1]\}, \ \sum_{i=1}^{2^m} \lambda_i = 1\} \ . \tag{12}$$

Alternatively we label the m different values of $\mathbf{g}$ as $\mathbf{g}_1, \mathbf{g}_2, ..\mathbf{g}_m$, and express these as a function

$$\mathbf{g}(\mu_1, \mu_2, ..\mu_m) = \sum_{p_1=0}^{1} \sum_{p_2=0}^{1} ... \sum_{p_m=0}^{1} \mu_1^{(p_1)} \mu_2^{(p_2)} ... \mu_m^{(p_m)} \mathbf{g}_{p_1 p_2 ... p_m} \ , \tag{13}$$

where each μ_j is either 0 or 1, and again using the shorthand $\mu_j^{(0)} = \mu_j$, $\mu_j^{(1)} = 1 - \mu_j$. Each control μ_j is then replaced with a variable $\lambda_j \in [0,1]$, and we define a canopy of $\mathbf{g}$ values,

$$\mathbf{g} \in \mathcal{G} = \left\{ \mathbf{g}(\lambda_1, \lambda_2, .., \lambda_m) : \begin{array}{ll} \lambda_j \in [0,1] & \text{for} \quad \mathbf{x} \in \mathcal{D} \\ \lambda_j = 0 \text{ or } 1 & \text{for} \quad \mathbf{x} \notin \mathcal{D} \end{array} \right\} \ , \tag{14}$$

in terms of the function (13).

To resolve the discontinuity we can now make one of two physically reasonable assumptions, that for $\mathbf{x} \in \mathcal{D}$:

$$\begin{array}{l} (a) \ \mathbf{f} \in \mathcal{F}, \text{ called the } \textit{field combination}; \\ (b) \ \mathbf{g} \in \mathcal{G}, \text{ called the } \textit{parametric combination}. \end{array} \tag{15}$$

Combination (15)(a) is the one considered in previous sections, and is the convex set usually associated with Filippov's differential inclusions [11]. Combination (15)(b) might be considered more general, because it describes a system with a known dependence on some switchable quantity $\mathbf{g}$, and is inspired by Utkin's equivalent control method, where $\mathbf{g}$ would be a control parameter. One might consider the two alternatives as viewpoints either of a controller or an observer. The controller knows that the field $\mathbf{f}$ depends

smoothly upon a particular function $\mathbf{g}$ in which all switching takes place, and so she applies the parametric combination (13) to $\mathbf{g}$. The impotent observer knows only the value of the field $\mathbf{f}$, so he applies the field combination by applying (6) to $\mathbf{f}$ in place of $\mathbf{g}$. These two approaches at least provide a reasonable deterministic system in the majority of cases.

To derive the dynamics at the discontinuity we seek solutions for the λ_j for which $\mathbf{g}_\star$ satisfies the set of sliding conditions (4), which become

$$\mathbf{f}(\mathbf{x}, \mathbf{g}_*) \cdot \nabla h_j(\mathbf{x}) = 0 \quad \text{for every} \quad j = 1, 2, ..., m \ .$$

As for the combination (15)(a) which we explored in the previous section, finding sliding vectors requires solving this set of m equations for the m unknowns $\lambda_1, \lambda_2, .., \lambda_m$, each within the range $[0, 1]$. So if solutions exist they will typically be isolated, and therefore finite in number.

The switching variables μ_j appear nonlinearly (i.e. as products of multiple μ_j's) in (13). In the setting of switched or variable structure electronic design [23], the system would usually depend linearly on a set of control variables. We can rewrite (13) in such a form by letting $\mathbf{g} = \mathbf{a} + B\mathbf{u}$ for some vector of m scalar controls u_j, $j = 1, 2, ..., m$, where B is an $N \times m$ matrix. For example, this is easily achieved by letting $u_j = p_j$. When we solve the sliding problem we obtain a particular set of u_j's called the "equivalent controls" u_j^{eq}.

In general, the set obtained from the parametric combination, $\mathbf{f}(\mathbf{x}, \mathcal{G})$, where $\mathcal{G}$ is a convex set of $\mathbf{g}_1, \mathbf{g}_2, ..., \mathbf{g}_{2^m}$, is not the same as the set obtained from the field combination, $\mathcal{F}$, where $\mathcal{F}$ is a convex set of $\mathbf{f}$ values $\mathbf{f}(\mathbf{x}, \mathbf{g}_1), \mathbf{f}(\mathbf{x}, \mathbf{g}_2), ..., \mathbf{f}(\mathbf{x}, \mathbf{g}_{2^m})$. When $\mathbf{f}$ depends linearly on $\mathbf{g}$, however, these sets coincide as $\mathcal{F} = \mathbf{f}(\mathbf{x}, \mathcal{G})$. In [9], a convex canopy of the form (7) is taken over field values $\mathbf{f}_{\mu_1, \mu_2, ...}$, but the result above shows how easily this is extended to a combination over input values $\mathbf{g}_{\mu_1, \mu_2, ...}$. This reveals some ambiguity in the argument of [8] that one particular combination can be justified over another by smoothing out the discontinuity. The alternatives of parametric or field combination imply that different choices of smoothing may in fact be *chosen* to justify any such convex combinations, and the ambiguity of such smoothing is discussed in [16]. Nevertheless, the two considered here are natural in the sense that they assume switches occur within sets of physically reasonable values, applied either to the vector field $\mathbf{f}$, or to some input vector $\mathbf{g}$ which appears as a parameter of $\mathbf{f}$.

7. Further examples: the intersection and novel attractors

We illustrate here how the canopy extension of the Filippov method leads to a classification of sliding regions at the intersection of switching manifolds. We specify vector fields $\mathbf{f}_i$ and use the field combination. We begin with three flows that resemble equilibria familiar from smooth systems, and conclude with something less familiar: a determinacy-breaking singularity.

In some cases, an intersection resembles an equilibrium in a smooth flow, with switching manifolds taking the role of attracting and repelling manifolds. As a result, sliding regions at the intersection of two manifolds may be of focus, node, or saddle type, or combinations thereof at the intersection of several manifolds.

To illustrate this, take coordinates $\mathbf{x} = (X, Y, \underline{Z})$, where the underline indicates that $\underline{Z}$ is a vector of dimension $N - 2$. Then consider the system

$$\dot{\mathbf{x}} = \{\mathbf{f}_i(\mathbf{x}) \text{ if } \mathbf{x} \in R_i\}_{i=00,10,01,11,}$$

where the four regions

$$
\begin{aligned}
R_{00} &= \left\{ \mathbf{x} \in \mathbb{R}^N : \ X, Y > 0 \right\} , \\
R_{01} &= \left\{ \mathbf{x} \in \mathbb{R}^N : \ X > 0, Y < 0 \right\} , \\
R_{11} &= \left\{ \mathbf{x} \in \mathbb{R}^N : \ X, Y < 0 \right\} , \\
R_{10} &= \left\{ \mathbf{x} \in \mathbb{R}^N : \ X < 0, Y > 0 \right\} ,
\end{aligned}
$$

are separated by switching manifolds

$$
\begin{aligned}
\mathcal{D}_1 &= \left\{ \mathbf{x} \in \mathbb{R}^N : \ h_1(X) := X = 0 \right\} , \\
\mathcal{D}_2 &= \left\{ \mathbf{x} \in \mathbb{R}^N : \ h_2(Y) := Y = 0 \right\} .
\end{aligned}
$$

The convex combinations across $\mathcal{D}_1$ and $\mathcal{D}_2$ are respectively

$$
\begin{aligned}
\mathbf{f}_{\star p} &= \lambda_1 \mathbf{f}_{0p} + (1 - \lambda_1)\mathbf{f}_{1p} && \text{on} \quad \mathcal{D}_1 , \\
\mathbf{f}_{p\star} &= \lambda_2 \mathbf{f}_{p0} + (1 - \lambda_2)\mathbf{f}_{p1} && \text{on} \quad \mathcal{D}_2 ,
\end{aligned}
$$

for $p = 0$ or $p = 1$. Sliding occurs on $\mathcal{D}_1$ if there exists a solution $\lambda_1 \in [0, 1]$ to the problem $\mathbf{f}_{\star p} \cdot \nabla h_1 = 0$, and is given by $\dot{\mathbf{x}} = \mathbf{f}_{\star p}$. Sliding likewise occurs on $\mathcal{D}_2$ if there exists a solution $\lambda_2 \in [0, 1]$ to the problem $\mathbf{f}_{p\star} \cdot \nabla h_2 = 0$, and is given by $\dot{\mathbf{x}} = \mathbf{f}_{p\star}$.

The convex combination across the intersection $\mathcal{D}_1 \cap \mathcal{D}_2$ is given by

$$\mathbf{f}_{\star\star} = \lambda_2[\lambda_1 \mathbf{f}_{00} + (1 - \lambda_1)\mathbf{f}_{10}] + (1 - \lambda_2)[\lambda_1 \mathbf{f}_{01} + (1 - \lambda_1)\mathbf{f}_{11}] , \tag{16}$$

and gives a sliding system $\dot{\mathbf{x}} = \mathbf{f}_{\star\star}$ if there exist solutions $\lambda_1 \in [0,1]$ and $\lambda_2 \in [0,1]$ to the simultaneous equations

$$\mathbf{f}_{\star\star} \cdot \nabla h_1 = 0 \qquad \text{and} \qquad \mathbf{f}_{\star\star} \cdot \nabla h_2 = 0 \; . \tag{17}$$

We will take examples of the $\mathbf{f}_i$ from four vector fields

$$\mathbf{f}_a = (1, \; -\alpha, \; \underline{a}) \; , \qquad \mathbf{f}_b = (-1, \; -\beta, \; \underline{b}) \; ,$$
$$\mathbf{f}_c = (-1, \; \gamma, \; \underline{c}) \; , \qquad \mathbf{f}_d = (1, \; \delta, \; \underline{d}) \; ,$$

where the underline denotes an $(n-2)$ dimensional vector (as in $x = (X, Y, \underline{Z})$), and where $\alpha, \beta, \gamma, \delta > 0$ are constants. Below we assign each of the labels a, b, c, d, to one of $00, 01, 10, 11$, then find any sliding dynamics, if it exists. There are numerous possible cases, we consider only a few that are indicative of the key ideas, beginning with the three shown in figure 8.
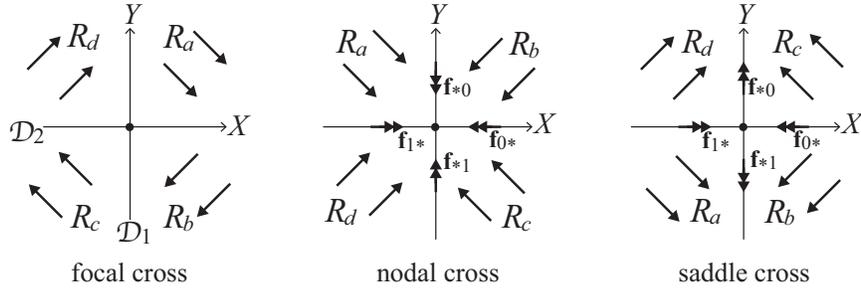

Figure 8: Three sliding geometries that form focus-like, node-like, and saddle-like, equilibria in the X-Y subsystem. The assignment of the regions $R_{a,b,c,d}$ are shown. The sliding vector fields $\mathbf{f}_{\star i}$ on $\mathcal{D}_1$ and $\mathbf{f}_{i\star}$ on $\mathcal{D}_2$ are labeled.

7.1. A focal cross

For the first case in figure 8 we let

$$a = 00 \; , \qquad b = 01 \; , \qquad c = 11 \; , \qquad d = 10 \; .$$

There is no sliding on the manifolds $\mathcal{D}_1$ and $\mathcal{D}_2$, except at their intersection. The convex combination (16) is found to contain a vector lying in $\mathcal{D}_1 \cap \mathcal{D}_2$ for $\lambda_1 = \frac{\gamma+\delta}{\alpha+\beta+\gamma+\delta}$, $\lambda_2 = 1/2$, giving

$$\mathbf{f}_{\star\star} = \frac{(\, 0 \,, \, 0 \,, \, (\alpha+\beta)\underline{c} + (\alpha+\beta)\underline{d} + (\gamma+\delta)(\underline{a}+\underline{b}) \,)}{2(\alpha+\beta+\gamma+\delta)} \; ,$$

and hence the unique sliding dynamics is given by $\dot{\mathbf{x}} = \mathbf{f}_{\star\star}$. The solutions for λ_1 and λ_2 lie in the range $[0,1]$ since $\alpha, \beta, \gamma, \delta > 0$, so the sliding vector $\mathbf{f}_{\star\star}$

always exists. The form of this dynamics within the intersection $X = Y = 0$ depends on the $N - 2$ dimensional vectors $\underline{a}$, $\underline{b}$, $\underline{c}$, $\underline{d}$, which we will not specify further here.

Outside the sliding region, the flow winds clockwise through the regions $R_{00}, R_{01}, R_{11}, R_{10}$, in order, crossing the boundaries $\mathcal{D}_1$ and $\mathcal{D}_2$ transversally. As it does, it spirals inward or outward with respect to the $N-2$ dimensional sliding region $X = Y = 0$. For this constant vector field, by integrating around the flow in the X-Y plane, it is easily found that the flow spirals inward to $X = Y = 0$ if $\beta\delta < \alpha\gamma$, and outward from it if $\beta\delta > \alpha\gamma$. The sliding region is therefore asymptotically attracting if $\beta\delta < \alpha\gamma$, and asymptotically repelling if $\beta\delta > \alpha\gamma$.

7.2. A nodal cross

The second case in figure 8 is given by setting

$$b = 00 , \qquad c = 01 , \qquad d = 11 , \qquad a = 10 .$$

The manifolds $\mathcal{D}_1$ and $\mathcal{D}_2$ consist entirely of sliding regions, with vector fields

$$\left. \begin{array}{l} \mathbf{f}_{\star 0} = (\, 0, \ -\alpha - \beta, \ \underline{a} + \underline{b} \,) \, / 2 \\ \mathbf{f}_{\star 1} = (\, 0, \ \gamma + \delta, \ \underline{c} + \underline{d} \,) \, / 2 \end{array} \right\} \qquad \text{on} \qquad \mathcal{D}_1 \, ,$$

$$\left. \begin{array}{l} \mathbf{f}_{0\star} = (\, -1, \ 0, \ (\beta\underline{c} + \gamma\underline{b}) \, / \, (\beta + \gamma) \,) \\ \mathbf{f}_{1\star} = (\, 1, \ 0, \ (\alpha\underline{d} + \delta\underline{a}) \, / \, (\alpha + \delta) \,) \end{array} \right\} \qquad \text{on} \qquad \mathcal{D}_2 \, ,$$

given by solving (17) to find $\lambda_1 = 1/2$ on $\mathcal{D}_1$ for both $\mathbf{f}_{\star 0}$ and $\mathbf{f}_{\star 1}$, while on $\mathcal{D}_2$ we have $\lambda_2 = \frac{\gamma}{\beta + \gamma}$ for $\mathbf{f}_{0\star}$ and $\lambda_2 = \frac{\delta}{\alpha + \delta}$ for $\mathbf{f}_{1\star}$. All of these solutions for λ_1 and λ_2 lie in the range $[0, 1]$ since $\alpha, \beta, \gamma, \delta > 0$, so the sliding vectors $\mathbf{f}_{i\star}$ and $\mathbf{f}_{\star i}$ always exists.

On the intersection $X = Y = 0$ we find a valid sliding vector field, with $\lambda_1 = 1/2$ and $\lambda_2 = (\gamma + \delta)/(\alpha + \beta + \gamma + \delta)$, which lie in the range $[0, 1]$ since $\alpha, \beta, \gamma, \delta > 0$. So the sliding vector $\mathbf{f}_{\star\star}$ always exists and is given by

$$\mathbf{f}_{\star\star} = \frac{(\, 0 \, , \ 0 \, , \ (\alpha + \beta)(\underline{c} + \underline{d}) + (\gamma + \delta)(\underline{a} + \underline{b}) \,)}{2(\alpha + \beta + \gamma + \delta)} \, .$$

In this node-like case, the flow is attracted (in finite time) to the switching manifolds $\mathcal{D}_1$ and $\mathcal{D}_2$, on which the sliding flow is then attracted (also in finite time) to the intersection $\mathcal{D}_1 \cap \mathcal{D}_2$, where further sliding dynamics ensues as given by $\dot{\mathbf{x}} = \mathbf{f}_{\star\star}$, determined by the higher dimensional constants $\underline{a}, \underline{b}, \underline{c}, \underline{d}$.

7.3. A saddle cross

For the third case in figure 8 we assign

$$c = 00 , \qquad b = 01 , \qquad a = 11 , \qquad d = 10 ,$$

As in the nodal case, the manifolds $\mathcal{D}_1$ and $\mathcal{D}_2$ consist entirely of sliding regions, now with vector fields

$$
\left.
\begin{array}{l}
\mathbf{f}_{\star 0} = (\, 0, \ \gamma + \delta, \underline{c} + \underline{d} \,) \, / 2 \\
\mathbf{f}_{\star 1} = (\, 0, \ -\alpha - \beta, \ \underline{a} + \underline{b} \,) \, / 2
\end{array}
\right\} \qquad \text{on} \quad \mathcal{D}_1 ,
$$

$$
\left.
\begin{array}{l}
\mathbf{f}_{0\star} = (\, -1, \ 0, \ (\beta \underline{c} + \gamma \underline{b}) \, / \, (\beta + \gamma) \,) \\
\mathbf{f}_{1\star} = (\, 1, \ 0, \ (\alpha \underline{d} + \delta \underline{a}) \, / \, (\alpha + \delta) \,)
\end{array}
\right\} \qquad \text{on} \quad \mathcal{D}_2 ,
$$

given by solving (17) to find $\lambda_1 = 1/2$ on $\mathcal{D}_1$, while on $\mathcal{D}_2$ we get $\lambda_2 = \frac{\beta}{\beta + \gamma}$ for $\mathbf{f}_{0\star}$ and $\lambda_2 = \frac{\alpha}{\alpha + \delta}$ for $\mathbf{f}_{1\star}$. Sliding on $\mathcal{D}_2$ flows towards the intersection $\mathcal{D}_1 \cap \mathcal{D}_2$, while sliding on $\mathcal{D}_1$ flows away from the intersection.

On the intersection we find the same sliding vector field as the nodal case, with $\lambda_1 = 1/2$ and $\lambda_2 = (\alpha + \beta)/(\alpha + \beta + \gamma + \delta)$, giving

$$\mathbf{f}_{\star\star} = \frac{(\, 0 , \ 0 , \ (\alpha + \beta)(\underline{c} + \underline{d}) + (\gamma + \delta)(\underline{a} + \underline{b}) \,)}{2(\alpha + \beta + \gamma + \delta)} .$$

As in previous examples, the solutions for λ_1 and λ_2 on $\mathcal{D}_1$, $\mathcal{D}_2$, and their intersection, all lie in the range $[0, 1]$ since $\alpha, \beta, \gamma, \delta > 0$, so the sliding vectors given above always exists.

The flow has a saddle-like topology around the intersection, where the manifolds $\mathcal{D}_1$ and $\mathcal{D}_2$ play the role of repelling and attracting manifolds respectively. The flow outside the threshold arrives at $\mathcal{D}_1$ or departs $\mathcal{D}_2$ in finite time. It then slides to arrive at the intersection $X = Y = 0$ along $\mathcal{D}_2$, or depart it along $\mathcal{D}_1$, also in finite time.

As a result, the flow through a point in this system is not unique for all time. Any point $\mathbf{x} \in \mathcal{D}_2$ has a non-unique flow, because while sliding it can leave $\mathcal{D}_2$ at any time up to its arrival at the intersection with $\mathcal{D}_1$. The sliding vector fields, however, are all well defined, and the dynamics can be understood in the following sense: every trajectory lies *on* $\mathcal{D}_1$ at some earlier time and lies *on* $\mathcal{D}_2$ at some later time, just as it would lie asymptotically *close* to $\mathcal{D}_1/\mathcal{D}_2$ at earlier/later times if they were repelling/attracting manifolds of an equilibrium in a smooth flow.

7.4. A 4-roll attractor

Consider a three-dimensional system that switches between vector fields

$$\begin{aligned}
\mathbf{f}_{00} &= (+1/10, \alpha Y - (Z-1), \alpha(Z-1) + Y) \ , \\
\mathbf{f}_{11} &= (-1/10, \alpha Y + (Z+1), \alpha(Z+1) - Y) \ , \\
\mathbf{f}_{10} &= (\alpha(X+1) - Y, \alpha Y + (X+1), +1/10) \ , \\
\mathbf{f}_{01} &= (\alpha(X-1) + Y, \alpha Y - (X-1), -1/10) \ ,
\end{aligned}$$

with switching functions $h_1 = Z + X$ and $h_2 = Z - X$. The discontinuity surfaces as usual are $\mathcal{D}_1$ at $h_1 = 0$, $\mathcal{D}_2$ at $h_2 = 0$, at which the first and second indices of $\mathbf{f}_{p_i p_j}$ flip respectively.

Sliding exists on subsets of the discontinuity surfaces:

- on $\{\mathcal{D}_1 : h_2 > 0, |\alpha(X+1) - Y| > 1/10\}$, with $\lambda_1 = \frac{1}{2} + \frac{1}{20(\alpha(X+1)-Y)}$,
- on $\{\mathcal{D}_1 : h_2 < 0, |\alpha(X-1) + Y| > 1/10\}$, with $\lambda_1 = \frac{1}{2} + \frac{1}{20(\alpha(X-1)+Y)}$,
- on $\{\mathcal{D}_2 : h_1 > 0, |\alpha(X-1) + Y| > 1/10\}$, with $\lambda_2 = \frac{1}{2} + \frac{1}{20(\alpha(X-1)+Y)}$,
- on $\{\mathcal{D}_2 : h_1 < 0, |\alpha(X+1) - Y| > 1/10\}$, with $\lambda_2 = \frac{1}{2} + \frac{1}{20(\alpha(X+1)-Y)}$,

with crossing elsewhere (the 'elsewhere' being thin strips of the form $|\alpha(x \pm 1) \mp y| < \frac{1}{10}$ that are the complement of the sliding sets on $\mathcal{D}_1$ and $\mathcal{D}_2$).

Sliding exists on whole intersection $X = Z = 0$ with $\lambda_1 = \lambda_2 = 1/2$, giving a sliding vector field

$$\dot{Y} = 1 + \alpha Y \ .$$

By calculating the sliding vector field on each discontinuity manifold $\mathcal{D}_1$ and $\mathcal{D}_2$ (which we omit here), one finds that they are directed towards the intersection for $Y < \alpha$ and away from it for $Y > \alpha$. This implies that, if a trajectory is ejected from the intersection, it will return there after a rotation through one the vector fields $\mathbf{f}_{00}, \mathbf{f}_{10}, \mathbf{f}_{01}, \mathbf{f}_{11}$, as sketched in figure 9. This implies that every trajectory in some two-dimensional set is funnelled through a common point along the intersection, lying near the origin, and constituting a *determinacy-breaking event* because it lies in both the past and future of a continuous family of different trajectories. The result is, for certain values of α, an attracting set consistent with a novel form of chaos that is driven by determinacy-breaking events, previously seen at certain singularities in piecewise-smooth systems [5] (usually dubbed *non-deterministic chaos* for short).

In figure 10 this system is simulated numerically in a similar manner to figure 7, by smoothing the system. The left figure shows a value of α for which the set sketched in figure 9 is an attractor, forming an invariant set
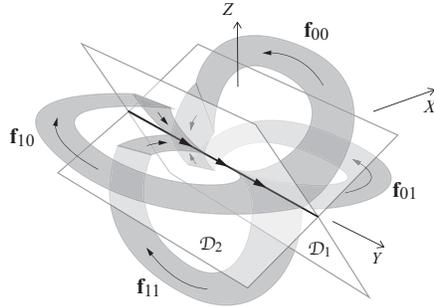
Figure 9: Sketch of the invariant set with a determinacy breaking event.

on which the simulation exhibits sensitive dependence on initial conditions. The right figure shows that other, deterministic, dynamics is possible in the system, at different values of α, this case involving three co-existing multiple-loop periodic orbits encircling the intersection with no sliding. One problem for future work is to study the bifurcations that take place between these figures, i.e. as α varies, and in general to study periodic bifurcations involving intersections.
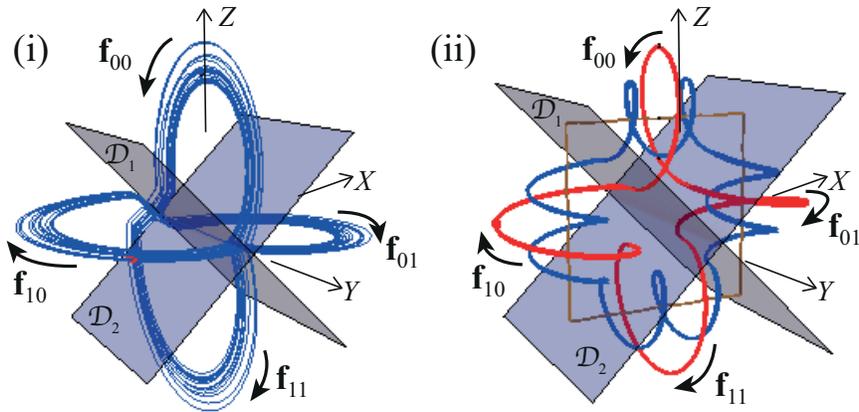

Figure 10: Attractors in the 4-roll system for: (i) $\alpha = 10^{-1}$, (ii) $\alpha = 10^{-4}$. The simulation is made by approximating each discontinuous λ_j by a sigmoid function $\lambda_j \mapsto [1 + \tanh(kh_j)]/2$ with $k = 10^6$, and solving the initial value problem numerically using Mathematica$^{\text{TM}}$ function NDSolve; this procedure leads to a complex attractor in (i), and three different attractors with 0, 4, or 8 rotations per period in (ii). Each attractor is shown by simulating one trajectory over a time $t = 1000$.

8. Closing remarks

The canopy construction and dummy system provided here propose a route to deterministic dynamics in a discontinuous system, extending Filippov's method by assuming simply that there is no bias across any of the discontinuities. We consider discontinuity surfaces consiting of transversally intersecting manifolds. In non-generic cases (e.g. when manifolds meet tangentially), to obtain a unique system one must typically impose further conditions specific to the context. The canopy construction makes a systematic study of the fixed points and periodic orbits at intersections possible.

There are three important problems raised in this paper which cannot be fully explored here, because more study is needed of the dynamical theory, particularly in the light of the growing number of applications to physics and biology. We conclude, therefore, with a brief review of these to set the scene for ongoing study.

8.1. Justification for the dummy system

It is obvious to ask the question: why choose the set we have dubbed the *canopy* (7), rather than accepting the lack of uniqueness that comes from (5)? If one is able to find physical laws for the dynamics on $\mathcal{D}$ then these supersede the discontinuous model (1), and whether these agree with the canopy and dummy dynamics is open for experimenters of various disciplines to put to the test. In many cases no such physical laws are known, then we have shown that the canopy is a natural choice in that, by treating each switch in the system without bias and assuming no inter-relation between vector fields, it prescribes deterministic dynamics under the least possible assumptions. We have further shown that it agrees with numerical simulations.

We have intentionally left open the question of whether there is a rigorous justification for the canopy and for Conjecture 4. On such justification is given in [16] for a single discontinuity, but in seeking such a justification it seems possible to take very different routes to arrive at the canopy combinations and dummy dynamics as defined here. Work in this direction, however, is still in progress. One may, for example, seek to derive the results above by applying Euler's method or some other discretization across the discontinuity (similar to [12]), by studying the small-limit of hysteresis at the discontinuity (as Filippov and Utkin did [11, 24]), by smoothing the discontinuity and considering it the limit of a regular perturbation problem (as in [1]) or singular perturbation problem (as in [16, 22]), or by considering small noise (as in [21]); all of which await further analysis where intersections are concerned. Forthcoming work by multiple authors seeks to add

rigour and generality to this, it can only be remarked here that the theory of perturbations of nonsmooth systems is one of increasing interest with many open questions.

Results such as those presented here suggest the possible routes and benefits for such a theory. For brevity, and to avoid biasing further investigation, the decision has been made to avoid lengthy derivations, instead making only the weaker case by numerical evidence in section 7 that smoothing reproduces the canopy result, and is sufficiently robust to numerical errors of discretization and computational noise to produce the results in figure 7.

An indication of the richness of questions opened up such questions is highlighted by section 6, where we show how the canopy combination can be applied in one of two ways: the parameteric or the field combination. A simple argument illustrates how different approaches might yield these different outcomes. One may consider that during an infinitesimal time δt spent on the threshold $\mathcal{D}$, the value of $\mathbf{f}$ lies close to each value $\mathbf{f}_{..0..}$ for a fraction λ_k of the time δt, and the opposite value $\mathbf{f}_{..1..}$ for the remaining fraction $1 - \lambda_k$. Breaking each of these instants up across a hierarchy of manifolds $\mathcal{D}_k$, we find that $\mathbf{f}$ is given by the combination (6). Making the same argument based on the values of $\mathbf{g}$ leads similarly to (13).

From heuristic arguments in [16] for a single discontinuity, it appears that certain perturbations favour the parametric combination while others favour the field combination, and that the boundary between them is sharp. Rigorous proofs of these by smoothing and stochastic approaches are made in forthcoming work.

8.2. Different forms of ambiguity

The canopy construction removes one of the possible sources of ambiguity in how to define sliding dynamics along a discontinuity surface, but does not exhaust them. In section 7 we saw two examples of flows that are determined only up to a set of allowed values in forward time. This situation arises at any repelling sliding region, irrespective of intersections. In Filippov's original concept, the history of the sliding region is likewise determined only up to a set of allowed values. Novel dynamics arises when set-valued histories meet with set-valued futures at intersections such as that in figure 9, and also at singularities where vector fields are tangent to both sides of a switching manifold, the simplest of which is the two-fold singularity [5]. We could attempt to remove any ambiguity by imposing higher order conditions to recover uniqueness, for example insisting that the flow through a singularity is smooth, but this is a strange choice when nearby

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there will still exist trajectories which can not be smoothed out, e.g. crossing trajectories as in figure 1(i). We could attempt to remove ambiguity by imposing mechanical or control-motivated constraints, but we give up the generality of Filippov's groundbreaking method, with which it continues to find novel applications. In certain situations not considered here, non-uniqueness actually gives rise to well-defined dynamics *within sets of allowed values*. This implies a breakdown of strict determinism that is inherent to the local geometry, and due to singularities.

An important message to impart is therefore as follows. One should not be over eager to impose uniqueness on the solutions of discontinuous differential equations if uniqueness is not implied. Situations are beginning to emerge in which lack of uniqueness is precisely what is required to characterize extreme sensitivity to initial conditions in the limit of discontinuity. Recent work on the breakdown of uniqueness at certain singularities leads one to consider discontinuity-induced *explosions* [17, 15] and their relation to canard explosions in singularly perturbed systems [6]. Loss of uniqueness in the sense of non-invertibility of the flow is also a familiar feature of frictional sticking (see e.g. [7]), and even repelling sliding is known to have physical applications [2, 15]. Thus although we have sought to extend deterministic dynamics as far as possible here, we have not entirely banished non-uniqueness from nonsmooth systems in general, and more importantly nor should we seek to do so.

8.3. Why try to construct unique trajectories at all?

The method given in this paper, for resolving discontinuities using convex combinations, goes beyond what Filippov described but aims to be consistent with it as far as possible. In many biological and engineering problems, we set out to model a deterministic system that exhibits some kind of switching, then find that discontinuities result in ambiguities when solving for the dynamics. Here we seek the least assumptions necessary to obtain unambiguous deterministic dynamics. The main assumption made is simply what we have described as isonomy between the different switches.

Finally, even with the canopy concept introduced here (building on work in [9]), we have seen that one may obtain finitely many alternative sliding modes, and while dummy dynamics determines which mode the dynamics will follow, the possible role of the different modes deserves ongoing study with an eye to applications.

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