ON SLIDING PERIODIC SOLUTIONS FOR PIECEWISE CONTINUOUS SYSTEMS DEFINED ON THE 2-CYLINDER

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ABSTRACT. This paper deals with discontinuous differential equations defined on the 2–dimensional cylinder. The main goal is to exhibit conditions for the existence of typical periodic solutions of such systems. An averaging method for computing sliding periodic solutions is developed, subject to convenient assumptions. We also apply the method to example problems. The main tools used are structural stability theory for discontinuous differential systems and Brouwer degree theory.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

Establishing the existence of periodic solutions is an important problem for understanding the dynamics of differential systems. Recently, degree theory has provided a considerable advance in the use of averaging methods to detect the existence of periodic solutions in dynamical systems (see for example [6], and for an introduction to the subject see [14, 15]). These advances assume that the differential systems involved are continuous.

In this paper, for a class of periodic differential equations with a discontinuity, an averaging method is used to establish the existence of period solutions. The theory of discontinuous systems has been developing at a fast pace in recent years, with growing importance at the frontier between mathematics, physics, engineering, and the life sciences. Interest stems particularly from discontinuous dynamical models in control theory [3], nonlinear oscillations [2, 13], impact and friction mechanics [5], economics [9, 10], biology [4], and others; a recent review appears in [12].

We discuss a special class of piecewise continuous differential systems defined on the cylinder. We consider only non-contractible periodic solutions, which are those wrapping around the cylinder such that they cannot be topologically contracted to a point, illustrated by type (b) rather than (a) in Figure 1. In a discontinuous system we can moreover distinguish between: orbits that never encounter the discontinuity; orbits that cross the discontinuity transversally (Figure
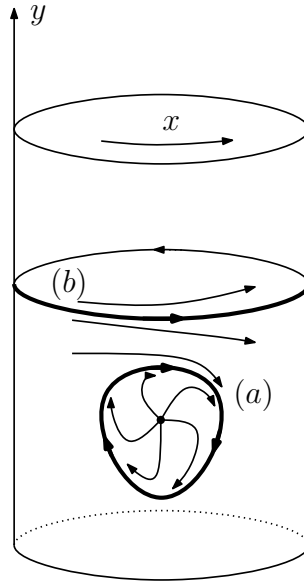
FIGURE 1.   Cylindrical phase portrait. Contractible (a) and non-
contractible (b) periodic solutions.

2); orbits that lie along the discontinuity for some time interval, termed *sliding*
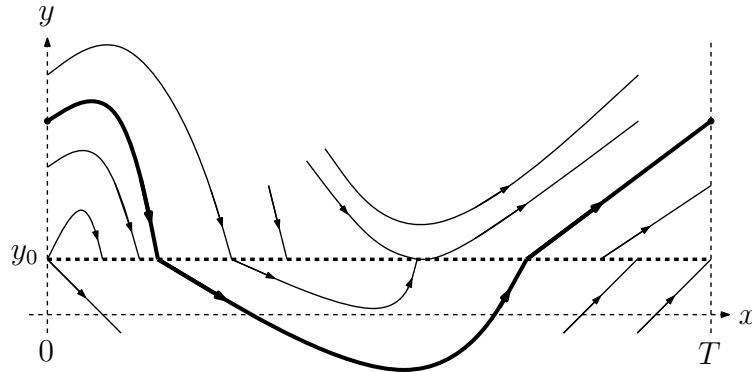(e.g. Figure 3); and orbits that involve both crossing and sliding (e.g. Figure 4).

FIGURE 2.   Non-contractible crossing periodic solution.

Crossing and sliding solutions are obtained by solving a discontinuous differ-
ential equation using the standard method of Filippov [8], summarized in the
appendix. Concisely, if a set of differential equations has a discontinuity at a
codimension one manifold Σ, the system may admit solutions either side of the
discontinuity that can be joined continuously, forming a solution that *crosses* Σ.
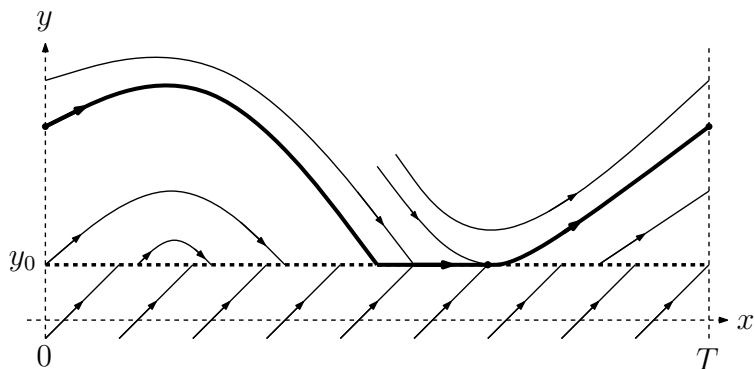
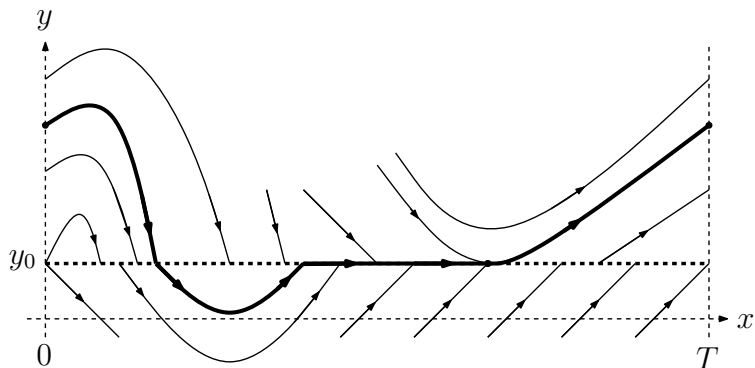FIGURE 3.   Non-contractible sliding periodic solution.

FIGURE 4.   Non-contractible crossing-sliding periodic solution.

Alternatively, solutions might be found to impinge upon Σ, after which they join continuously to solutions that *slide* inside Σ. The equations of these sliding solutions are given by a linear combination of the differential equations immediately either side of the discontinuity that lies tangent to the locus of discontinuity. Relevant details on these standard definitions (see e.g. [7, 8]) are summarized in the Appendix.

1.1. **Statement of the main results.** In [11] the methods of averaging theory for studying non-contractible crossing periodic solutions were extended to a class of discontinuous piecewise differential systems. The averaging methods relate solutions of certain systems of non-autonomous differential equations to solutions of algebraic equations. In this paper we extend the method to detect periodic solutions that involve sliding at a discontinuity. The main tool used is *Brouwer Degree Theory*, which is defined in Section 2.

The piecewise continuous systems of interest can be written in the form of the differential equation

$$(1) \qquad\qquad y'(x) = G(x, y) + \varepsilon F(x, y) + \varepsilon^2 R(x, y, \varepsilon),$$

where

$$G(x, y) = F_3(x, y) - \text{sign}(y - y_0)F_3(x, y),$$
$$F(x, y) = F_1(x, y) + \text{sign}(y - y_0)F_2(x, y),$$
$$R(x, y, \varepsilon) = R_1(x, y, \varepsilon) + \text{sign}(y - y_0)R_2(x, y, \varepsilon).$$

Here, $F_1, F_2, F_3 : \mathbb{R} \times D \to \mathbb{R}^2$ and $R_1, R_2 : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^2$ are continuous functions that are T-periodic in x and locally Lipschitz continuous, and D is an open interval of $\mathbb{R}$. The sign function creates a discontinuity at a set

$$(2) \qquad\qquad \Sigma := \{y = y_0, x \in \mathbb{R}\}.$$

We will consider systems for which one incidence of crossing and one interval of sliding are possible per period. We assume the geometry illustrated in Figure 5, which satisfies the following five conditions:

(A1) $F_1(\check{x}, y_0) + F_2(\check{x}, y_0) = 0$, $F_1(\check{x}, y_0) - F_2(\check{x}, y_0) > 0$ and $\partial_x F_1(\check{x}, y_0) + \partial_x F_2(\check{x}, y_0) < 0$;

(A2) $F_1(\bar{x}, y_0) + F_2(\bar{x}, y_0) = 0$, $F_1(\bar{x}, y_0) - F_2(\bar{x}, y_0) > 0$ and $\partial_x F_1(\bar{x}, y_0) + \partial_x F_2(\bar{x}, y_0) > 0$;

(A3) $F_1(x, y_0) + F_2(x, y_0) < 0$, for every $x \in (\check{x}, \bar{x})$;

(A4) $F_1(x, y_0) + F_2(x, y_0) > 0$, for every $x \in (\bar{x}, \check{x} + T)$;

(A5) $F_3(x, y_0) \geq 0$, for $x \in (\check{x}, \bar{x})$; or $F_3(x, y_0) \equiv 0$ and $F_1(x, y_0) - F_2(x, y_0) \geq 0$, for $x \in (\check{x}, \bar{x})$.
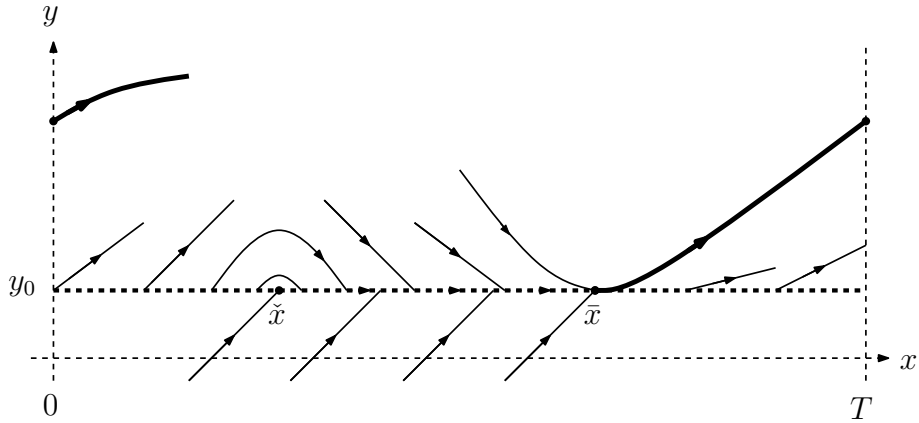

FIGURE 5. The geometry of the class of systems we consider, which allows one crossing and one sliding interval per period.

Proposition 1. *The class of functions defined by Hypotheses A1-A5 is not empty.*

Proof. We show this by constructing an example system satisfying $A1$-$A5$. Consider $F_3(x,y) = 0$, $F_1(x,y) = f_1(x)g_1(y)+h_1(y)$ and $F_2(x,y) = f_2(x)g_2(y)+h_2(y)$, where $y_0 = 0$, $g_1(0) = g_2(0) = 1$, $h_1(0) = h_2(0) = 0$, and

$$f_1(x) = 2 + \cos(x)\,, \qquad f_2(x) = -1 + \frac{1}{\sqrt{2}} - \sin\left(x - \frac{\pi}{4}\right).$$

Note that for every $x \in [0, 2\pi]$,

$$F_1(x,0) = f_1(x) = 2 + \cos(x) > -1 + \frac{1}{\sqrt{2}} - \sin\left(x - \frac{\pi}{4}\right) = f_2(x) = F_2(x,0).$$

Let $\check{x} = 3\pi/4$ and $\bar{x} = \pi$. Then

$$F_1(\check{x},0) + F_2(\check{x},0) = 0 \quad \text{and} \quad \partial_x F_1(\check{x},0) + \partial_x F_2(\check{x},0) = -\frac{1}{\sqrt{2}} < 0,$$

which satisfies condition $A1$, and

$$F_1(\bar{x},0) + F_2(\bar{x},0) = 0 \quad \text{and} \quad \partial_x F_1(\check{x},0) + \partial_x F_2(\check{x},0) = \frac{1}{\sqrt{2}} > 0,$$

which satisfies condition $A2$. Direct calculations then show that conditions $A3$-$A5$ are also satisfied. $\qquad\square$

Clearly (1) does not admit contractible periodic solutions. This can be seen by expressing it as an autonomous system

(3)
$$\dot{x}(t) = 1,$$
$$\dot{y}(t) = G(x,y) + \varepsilon F(x,y) + \varepsilon^2 R(x,y,\varepsilon),$$

where the dot denotes derivative with respect to t, then since $\dot{x}(t) = 1$, solutions can form closed curves only if they wrap around the cylinder. The main result of this paper concerns the existence of non-contractible sliding periodic solutions of system (1).

To state the main result, we first define the *Averaged Function* $f_0 : (\check{x} + T, \bar{x} + T) \to \mathbb{R}$, satisfying

(4)
$$f_0(z) = \int_{\bar{x}}^z F_1(s, y_0) + F_2(s, y_0)ds.$$

We then have the following theorem.

Theorem A. *Suppose that for some $a \in (\check{x} + T, \bar{x} + T)$ with $f_0(a) = 0$, there exists a neighborhood V of a such that $d_B(f_0, V, 0) \neq 0$, where d_B denotes the Brouwer degree (defined in Section 2). Then, for $\varepsilon > 0$ sufficiently small, there exists a non-contractible sliding T-periodic solution of system (1).*

Remark 1. *When f_0 is a C^1 function we can replace the Brouwer degree hypothesis by assuming $f_0'(a) \neq 0$.*

2. APPLICATION

Theorem A allows the study of autonomous planar piecewise continuous system with two zones when the set of discontinuity is a sphere, i.e. $\Sigma = h^{-1}(0)$ where $h(u, v) = u^2 + v^2 - r^2$. Here we give a few examples where the vector field switches between smooth functions X and Y either side of the discontinuity, corresponding to $y' = \varepsilon(F_1 + F_2) + \varepsilon^2(R_1 + R_2)$ and $y' = 2F_3 + \varepsilon(F_1 - F_2) + \varepsilon^2(R_1 - R_2)$ in appropriate coordinates (which are found in Section 5).

2.1. **Example 1.** Let $X(u, v)$ and $Y(u, v)$ be defined as

$$X(u, v) = (-v - \varepsilon au , \, u + \varepsilon b)$$

$$\text{and} \quad Y(u, v) = (u - v , \, u + v)$$

where a, b and ε are real parameters with $\varepsilon > 0$. Note that $X(u, v)$ is a linear center perturbed by an affine function, and $Y(u, v)$ is a focus. For $r = 1$, consider the piecewise continuous planar system

$$(5) \qquad (\dot{u}(t), \dot{v}(t)) = Z(u, v) = \begin{cases} X(u, v) & \text{if } h(u, v) > 0, \\ Y(u, v) & \text{if } h(u, v) < 0. \end{cases}$$

Proposition E1. *Assume that $a, b > 0$. Then, for $\varepsilon > 0$ sufficiently small, the piecewise continuous differential system (5) has a sliding periodic solution.*

For the proof of this theorem see Section 5.

2.2. **Example 2.** Let $X(u, v)$ and $Y(u, v)$ be defined as

$$X(u, v) = (-v + \varepsilon(au + buv) , \, u + \varepsilon(av + bv^2))$$

$$\text{and} \quad Y(u, v) = (-v + \varepsilon u^3 , \, u + \varepsilon v^3)$$

where a, b and ε are real parameters with $\varepsilon > 0$. Note that $X(u, v)$ is a linear center perturbed by a quadratic function, and $Y(u, v)$ linear center perturbed by a cubic function. For $r = 1$, consider the piecewise continuous planar system

$$(6) \qquad (\dot{u}(t), \dot{v}(t)) = Z(u, v) = \begin{cases} X(u, v) & \text{if } h(u, v) > 0, \\ Y(u, v) & \text{if } h(u, v) < 0. \end{cases}$$

Proposition E2. *Assume $-b < a < 0$. Then, for $\varepsilon > 0$ sufficiently small, the piecewise continuous differential system (6) has a sliding periodic solution.*
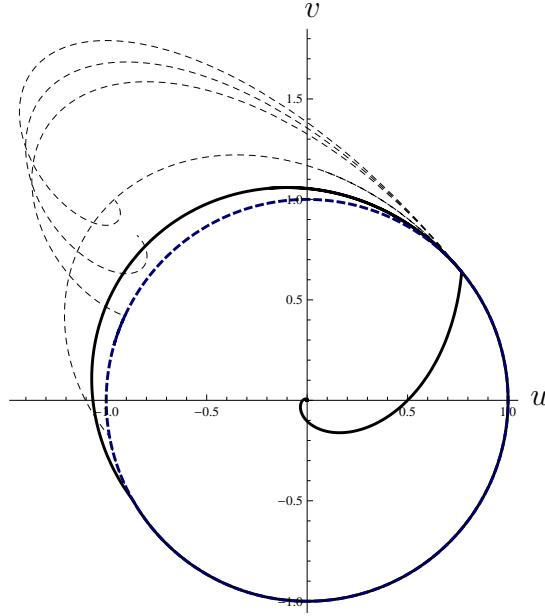
FIGURE 6. Numerical simulation of a solution of (5) with $a = b = 1$ passing through $(u, v) = (0.5, 0)$. The dashed lines indicate the solutions for $\varepsilon = 1; 0.9; 0.8; 0.35$; the non dashed bold line indicates the solution for $\varepsilon = 0.1$; and the dashed bold line indicates the set of discontinuity. For $\varepsilon = 1$ and $\varepsilon = 0.9$ the solution is a heteroclinic orbit, for the other values of ε the solution is a sliding periodic orbit.

For the proof of this theorem see Section 5.

The remainder of this paper sets out the proofs of the preceding results. Some necessary results on Brouwer degree are summarized in section 3, with important details regarding structural stability left to the Appendix. Theorem A is proven in section 4, followed by the proofs of Propositions E1 and E2 in section 5.

3. BASIC RESULTS ON BROUWER DEGREE

In this section, following [3], we present some results that we shall need for proving Theorem A. The following two theorems are proven in [3], and d_B is known as the Brouwer degree.

Theorem 2. *Let $P = \mathbb{R}^n = Q$ for a given positive integer n. For bounded open subsets V of P, consider continuous mappings $f : \overline{V} \to Q$, and points y_0 in Q such that y_0 does not lie in $f(\partial V)$ where ∂V denotes the boundary of V. Then to each such triple (f, V, y_0), there corresponds an integer $d_B(f, V, y_0)$ having the following three properties.*
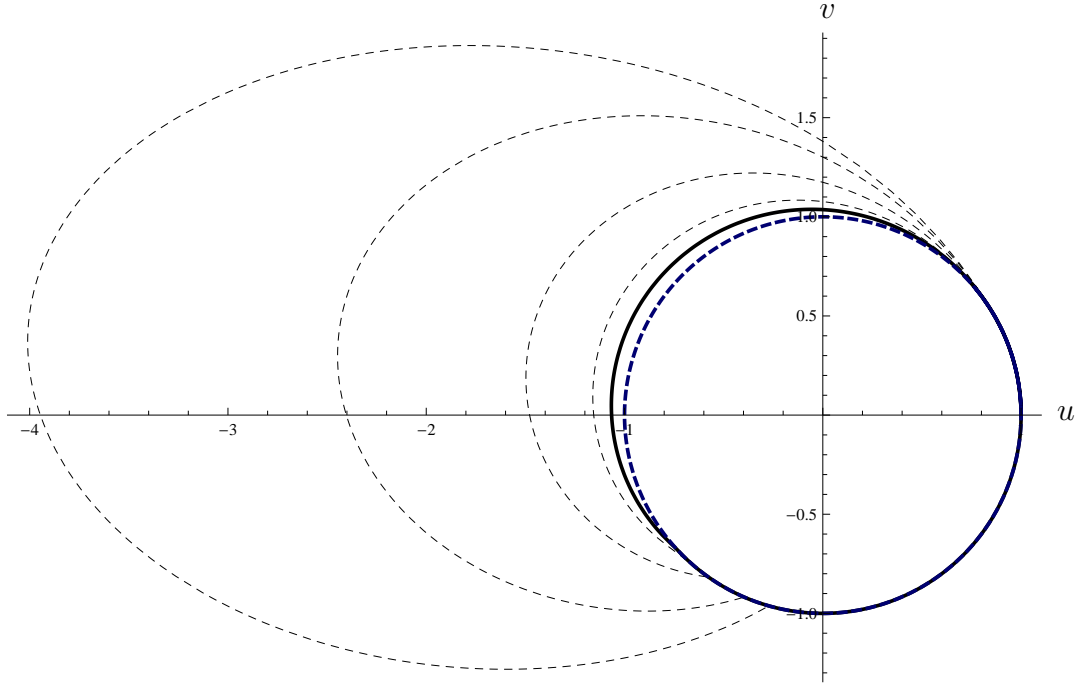
FIGURE 7.   Numerical simulation of a solution of (11) with $a = -1$ and $b = 2$ passing through $(u, v) = (1, 0)$. The dashed lines indicate the solutions for $\varepsilon = 0.35; 0.3; 0.2; 0.1$; the non dashed bold line indicates the solution for $\varepsilon = 0.05$; and the dashed bold line indicates the set of discontinuity.

(i) If $d_B(f, V, y_0) \neq 0$, then $y_0 \in f(V)$. If f_0 is the identity map of P onto Q, then for every bounded open set V and $y_0 \in V$, we have
$$d\left(f_0\big|_V, V, y_0\right) = \pm 1.$$

(ii) (*Additivity*) If $f : \overline{V} \to Q$ is a continuous map with V a bounded open set in P, and V_1 and V_2 are a pair of disjoint open subsets of V such that
$$y_0 \notin f(\overline{V} \backslash (V_1 \cup V_2)),$$
then,
$$d\left(f_0, V, y_0\right) = d\left(f_0, V_1, y_0\right) + d\left(f_0, V_1, y_0\right).$$

(iii) (*Invariance under homotopy*) Let V be a bounded open set in P, and consider a continuous homotopy $\{f_t : 0 \leq t \leq 1\}$ of maps of $\overline{V}$ in to Q. Let $\{y_t : 0 \leq t \leq 1\}$ be a continuous curve in Q such that $y_t \notin f_t(\partial V)$ for any $t \in [0, 1]$. Then $d_B(f_t, V, y_t)$ is constant in t on the interval $[0, 1]$.

Theorem 3. *The degree function $d_B(f, V, y_0)$ is uniquely determined by the conditions of Theorem 2.*

For the proof of the following Lemma refer to *Lemma 2.1* in [6].

Lemma 4. *Consider the continuous functions* $f_i : \overline{V} \to \mathbb{R}^n$, *for* $i = 0, 1, \cdots, k$, *and* $f, g, r : \overline{V} \times [\varepsilon_0, \varepsilon_0] \to \mathbb{R}^n$, *given by*

$$g(\cdot, \varepsilon) = f_1(\cdot) + \varepsilon f_2(\cdot) + \varepsilon^2 f_3(\cdot) + \cdots + \varepsilon^{k-1} f_k(\cdot),$$

$$f(\cdot, \varepsilon) = g(\cdot, \varepsilon) + \varepsilon^k r(\cdot, \varepsilon).$$

Assume that $g(z, \varepsilon) \neq 0$ *for all* $z \in \partial V$ *and* $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$. *Then, for* $|\varepsilon| > 0$ *sufficiently small,* $d_B(f(\cdot, \varepsilon), V, 0)$ *is well defined and*

$$d_B(f(\cdot, \varepsilon), V, 0) = d_B(g(\cdot, \varepsilon), V, 0).$$

4. PROOF OF THEOREM A

We denote

$$X(x, y, \varepsilon) = \Big(1, \quad \varepsilon(F_1(x, y) + F_2(x, y)) + \varepsilon^2(R_1(x, y, \varepsilon) + R_2(x, y, \varepsilon))\Big),$$

$$Y(x, y, \varepsilon) = \Big(1, \quad 2F_3(x, y) + \varepsilon(F_1(x, y) - F_2(x, y))$$
$$+ \varepsilon^2(R_1(x, y, \varepsilon) - R_2(x, y, \varepsilon))\Big),$$

and

$$h(x, y) = y - y_0.$$

Then equation (1) can be written as the autonomous system

$$(7) \qquad (\dot{x}(t), \dot{y}(t)) = Z(x, y) = \begin{cases} X(x, y) & \text{if } h(x, y) > 0, \\ Y(x, y) & \text{if } h(x, y) < 0, \end{cases}$$

which we denote concisely by $Z = (X, Y)_\Sigma$, with Σ denoting the discontinuity set (2). Z is defined on the cylinder $\mathbb{R} \times D$ embedded in $\mathbb{R}^3$, where D is an open interval of $\mathbb{R}$. Local solutions of (7) which pass through a point $p \in \Sigma$ are given by the Filippov convention (see Appendix, following the standard method of [8]).

Lemma 5. *For any* $\varepsilon > 0$ *sufficiently small there exist* $\check{x}_\varepsilon, \bar{x}_\varepsilon \in [0, T)$ *with* $\check{x}_\varepsilon \to \check{x}$ *and* $\bar{x}_\varepsilon \to \bar{x}$ *when* $\varepsilon \to 0$, *such that*

(a) $Xh(\check{x}_\varepsilon, y_0) = 0$, $X^2 h(\check{x}_\varepsilon, y_0) < 0$ *and* $Y(\check{x}_\varepsilon, y_0) > 0$;
(b) $Xh(\bar{x}_\varepsilon, y_0) = 0$, $X^2 h(\bar{x}_\varepsilon, y_0) > 0$ *and* $Y(\bar{x}_\varepsilon, y_0) > 0$;
(c) $Xh(x, y_0, \varepsilon) < 0$ *for every* $x \in (\check{x}_\varepsilon, \bar{x}_\varepsilon)$;
(d) $Xh(x, y_0, \varepsilon) > 0$ *for every* $x \in (\bar{x}_\varepsilon, \check{x}_\varepsilon + T)$; *and*
(e) $Yh(x, y_0, \varepsilon) \geq 0$ *for every* $x \in (\check{x}_\varepsilon, \bar{x}_\varepsilon)$.

Lemma 5 implies that the configuration of the cylindrical phase portrait of the system (1) is as given in Figure 8.
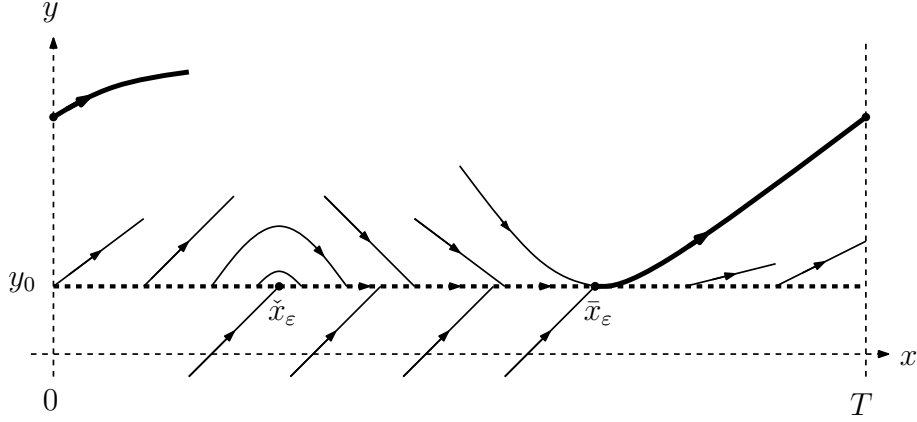
FIGURE 8. The geometry of the perturbed system, showing the points $\check{x}_\varepsilon$ and $\bar{x}_\varepsilon$.

Proof of Lemma 5. For simplicity we consider $F_3(x,y) \equiv 0$. When $F_3(x,y) \not\equiv 0$, the proof will follow similarly. For the discontinuous vector field $\hat{Z} = (\hat{X}, \hat{Y})_\Sigma$ corresponding to (1) we have

$$\hat{X}(x,y) = (1,\, F_1(x,y) + F_2(x,y)),$$
$$\hat{Y}(x,y) = (1,\, F_1(x,y) - F_2(x,y)).$$

Hypothesis A1 implies that $\hat{X}h(\check{x}, y_0) = 0$, $\hat{X}^2 h(\check{x}, y_0) < 0$, and $\hat{Y}h(\check{x}, y_0) > 0$. So $(\check{x}, y_0)$ is a points where the vector field $\hat{X}$ is tangent to Σ and curving towards it (an *invisible regular-fold point*, see Appendix). Analogously, hypothesis A2 implies that $\hat{X}h(\bar{x}, y_0) = 0$, $\hat{X}^2 h(\bar{x}, y_0) > 0$, and $\hat{Y}h(\bar{x}, y_0) > 0$. So $(\bar{x}, y_0)$ is a point where $\hat{X}$ is tangent to Σ and curving away from it (a *visible regular-fold point*, see Appendix). Therefore both $\check{x}$ and $\bar{x}$ on Σ are generic singularities of the discontinuous vector field $\hat{Z}$, which implies that $\hat{Z}$ is locally Σ-structurally stable at the points $(\check{x}, y_0)$ and $(\bar{x}, y_0)$ (see Appendix).

Now, we take the perturbation

(8) $$\hat{Z}_\varepsilon(x,y) = \left(\hat{X}_\varepsilon, \hat{Y}_\varepsilon\right)_\Sigma = \hat{Z}(x,y) + \varepsilon W(z,y,\varepsilon),$$

where

$$W(x,y) = \Big((1,\, R_1(x,y,\varepsilon) + R_2(x,y,\varepsilon))\,,\, (1,\, R_1(x,y,\varepsilon) - R_2(x,y,\varepsilon))\Big)_\Sigma.$$

For system (8), the existence of two regular-folds points $(\check{x}_\varepsilon, y_0)$ and $(\bar{x}_\varepsilon, y_0)$, respectively invisible and visible for $\hat{X}_\varepsilon$, is assured by the locally Σ-structurally stability (see Appendix). Moreover, $\check{x}_\varepsilon \to \check{x}$ and $\bar{x}_\varepsilon \to \bar{x}$, as $\varepsilon \to 0$. Since

$Z(x, y) = \varepsilon \hat{Z}_\varepsilon(x, y)$, it follows that, for $\varepsilon > 0$, $(\check{x}_\varepsilon, y_0)$ and $(\bar{x}_\varepsilon, y_0)$ are regular-folds points, respectively invisible and visible, of X in the system (7). Hence, items (a) and (b) of lemma have been proved.

Hypothesese A1 and A2 ensure the existence of $\kappa > 0$ such that

(i) $\hat{X}(x, y_0) < 0$ for every $x \in (\check{x}, \check{x} + \kappa)$ and $\hat{Y}(x, y_0) > 0$ for every $x \in (\bar{x} - \kappa, \bar{x})$;

(ii) $\hat{X}(x, y_0) > 0$ for every $x \in (\check{x} - \kappa, \check{x})$ and $\hat{Y}(x, y_0) > 0$ for every $x \in (\bar{x}, \bar{x} + \kappa)$.

To prove item (c) of the lemma we assume that there exist: a decreasing sequence $(\varepsilon_i)_{i \in \mathbb{N}} > 0$ such that $\varepsilon_i \to 0$; and a sequence $(z^i)_{i \in \mathbb{N}}$ such that $z^i \in (\check{x}_{\varepsilon_i}, \bar{x}_{\varepsilon_i})$, and $\hat{X}_{\varepsilon_i} h(z^i, y_0, \varepsilon_i) \geq 0$ for $i \in \mathbb{N}$. Observe that, since $Z = \varepsilon \hat{Z}_\varepsilon$, these assumptions are the contra-position of item (c).

From (i), we can choose conveniently a decreasing sequence $(\kappa_i)_{i \in \mathbb{N}} > 0$ such that $\kappa_i \to \kappa$, $z^i \in [\check{x}_{\varepsilon_i} + \kappa_i, \bar{x}_{\varepsilon_i} - \kappa_i]$, $\hat{X}_{\varepsilon_i} h(\check{x}_{\varepsilon_i} + \kappa_i, y_0, \varepsilon_i) < 0$, and $\hat{X}_{\varepsilon_i} h(\bar{x}_{\varepsilon_i} + \kappa_i, y_0, \varepsilon_i) < 0$, for $i \in \mathbb{N}$. Since $\hat{X}_{\varepsilon_i} h(z^i, y_0, \varepsilon_i) \geq 0$, it follows that there exists a sequence $(x^i)_{i \in \mathbb{N}}$ such that $x^i \in [\check{x}_{\varepsilon_i} + \kappa_i, \bar{x}_{\varepsilon_i} - \kappa_i]$, and $\hat{X}_{\varepsilon_i} h(x^i, y_0, \varepsilon_i) = 0$ for $i \in \mathbb{N}$. Therefore we can choose $i_0 \in \mathbb{N}$ sufficiently larger such that $(x^i)_{i \geq i_0} \subset [\check{x} + \kappa, \bar{x} - \kappa]$. So there is a convergent sub-sequence $(x^{i_j})_{j \in \mathbb{N}} \subset (\check{x}, \bar{x})$ such that $x^{i_j} \to x^0 \in (\check{x}, \bar{x})$. Thus

$$\begin{aligned} 0 = & \ \hat{X}_{\varepsilon_i} h(x^i, 0, \varepsilon_i) \\ = & \ \left(F_1(x^i, 0) - F_2(x^i, 0)\right) + \varepsilon_i \left(R_1(x^i, 0, \varepsilon) - R_2(x^i, 0, \varepsilon)\right), \end{aligned}$$

which implies that

$$F_1(x^{i_j}, 0) + F_2(x^{i_j}, 0) = -\varepsilon_{i_j} \left(R_1(x^{i_j}, 0, \varepsilon) + R_2(x^{i_j}, 0, \varepsilon)\right).$$

Applying the limit for $j \to \infty$ we obtain

$$(9) \qquad\qquad F_1(x^0, 0) + F_2(x^0, 0) = 0.$$

The expression (9) contradicts the Hypothesis A3. Therefore $\hat{X}_{\varepsilon_i} h(x, y_0, \varepsilon) < 0$ for every $x \in (\check{x}_\varepsilon, \bar{x}_\varepsilon)$. Since $X(x, y, \varepsilon) = \varepsilon \hat{X}_\varepsilon(x, y, \varepsilon)$, for $\varepsilon > 0$, we have that $Xh(x, y_0, \varepsilon) < 0$ for every $x \in (\check{x}_\varepsilon, \bar{x}_\varepsilon)$. Hence item (c) of lemma is proved. The proofs of items (d) and (e) follow similarly. $\qquad\square$

Lemma 6. *Let $f : [\bar{x}_\varepsilon, \infty) \to \mathbb{R}$ be the function defined as*

$$\begin{aligned} f(z) = & \int_{\bar{x}_\varepsilon}^{z} F_1(s, \varphi_X^2(s, \bar{x}_\varepsilon, \bar{x}_\varepsilon, y_0, \varepsilon)) + F_2(s, \varphi_X^2(s, \bar{x}_\varepsilon, \bar{x}_\varepsilon, y_0, \varepsilon))ds \\ & + \varepsilon \int_{\bar{x}_\varepsilon}^{z} R_1(s, \varphi_X^2(s, \bar{x}_\varepsilon, \bar{x}_\varepsilon, y_0, \varepsilon), \varepsilon) + R_2(s, \varphi_X^2(s, \bar{x}_\varepsilon, \bar{x}_\varepsilon, y_0, \varepsilon), \varepsilon)ds, \end{aligned}$$

where $\varphi_X(t, t_0, x_0, y_0, \varepsilon) = (\varphi_X^1(t, t_0, x_0, y_0, \varepsilon), \varphi_X^2(t, t_0, x_0, y_0, \varepsilon))$ is the flow induced by the vector field X, such that $\varphi_X(t_0, t_0, x_0, y_0, \varepsilon) = (x_0, y_0)$. Then, for

each $\varepsilon > 0$, *the system* (7) *admits a sliding periodic orbit if and only if there exists* x_ε, *with* $\check{x}_\varepsilon + T < x_\varepsilon \le \bar{x}_\varepsilon + T$, *such that* $f(x_\varepsilon) = 0$.

Proof. Let $Z_s(p)$ be the sliding vector field, defined (see Appendix) for $p \in \Sigma^s = \{(x, y_0) : x \in (\check{x}_\varepsilon, \bar{x}_\varepsilon)\}$, as

$$Z_s(p) = \frac{(Yh)(p)X(p) - (Xh)(p)Y(p)}{(Yh)(p) - (Xh)(p)}.$$

For system (7) this gives $Z_s \equiv (1,0)$, which implies that the solution passing through any $p \in \Sigma^s$ reaches $\bar{x}_\varepsilon$ in a finite time. If the solution passing through $\bar{x}_\varepsilon$ returns to Σ then it necessarily returns to Σ^s, providing in this way a sliding periodic solution. Therefore, system (7) admits a sliding periodic solution if and only if the solution passing through $\bar{x}_\varepsilon$ (bold line in Figure 8) returns to Σ.

We note that the solution starting at the point $(\bar{x}_\varepsilon, y_0)$ follows the flow of the vector field X. For $\varepsilon > 0$ sufficiently small, $\varphi_X(t, t_0, x, y, \varepsilon)$ is defined for every $t \in [0, T]$. Moreover

$$\varphi_X(t, t_0, x_0, y_0, \varepsilon) = (x_0, y_0) + \int_{t_0}^t X\left(\varphi_X(s, t_0, x_0, y_0, \varepsilon), \varepsilon\right) ds$$

So $\varphi_X^1(t, \bar{x}_\varepsilon, \bar{x}_\varepsilon, y_0, \varepsilon) = t$, and for $f(t) = \left(\varphi_X^2(t, \bar{x}_\varepsilon, \bar{x}_\varepsilon, y_0, \varepsilon) - y_0\right)/\varepsilon$ we have

$$f(t) = \int_{\bar{x}_\varepsilon}^t F_1(s, \varphi_X^2(s, \bar{x}_\varepsilon, \bar{x}_\varepsilon, y_0, \varepsilon)) + F_2(s, \varphi_X^2(s, \bar{x}_\varepsilon, \bar{x}_\varepsilon, y_0, \varepsilon)) ds$$

$$+ \varepsilon \int_{\bar{x}_\varepsilon}^t R_1(s, \varphi_X^2(s, \bar{x}_\varepsilon, \bar{x}_\varepsilon, y_0, \varepsilon), \varepsilon) + R_2(s, \varphi_X^2(s, \bar{x}_\varepsilon, \bar{x}_\varepsilon, y_0, \varepsilon), \varepsilon) ds,$$

Thus system (7) admits a sliding periodic solution if and only if there exists x_ε, with $\check{x}_\varepsilon + T < x_\varepsilon < \bar{x}_\varepsilon + T$, such that $\varphi_X^2(x_\varepsilon, \bar{x}_\varepsilon, \bar{x}_\varepsilon, y_0, \varepsilon) = y_0$, i.e. $f(\bar{x}_\varepsilon) = 0$. □

Proof of Theorem A. If A and B are open bounded intervals of $\mathbb{R}$ such that $[\bar{x}, \bar{x} + T] \subset A$ and $\bar{x} \in B$, then it is clear that for $\varepsilon_0 > 0$ sufficiently small $[\bar{x}_\varepsilon, \bar{x}_\varepsilon + T] \subset A$ and $\bar{x}_\varepsilon \in B$. By continuity of the application $\varphi_X^2(t, t_0, x_0, y_0, \varepsilon)$ and by compactness of the set $\overline{A} \times \overline{B}^2 \times \{y_0\} \times [-\varepsilon_0, \varepsilon_0]$, there exits K a compact subset of D such that $\varphi_X^2(t, t_0, x_0, y_0, \varepsilon) \in K$ for all $(t, t_0, x_0, y_0, \varepsilon) \in \overline{A} \times \overline{B}^2 \times \{y_0\} \times [-\varepsilon_0, \varepsilon_0]$. Now, by the continuity of the function R, $|R(t, \varphi_X^2(t, \bar{x}_\varepsilon, \bar{x}_\varepsilon, y_0, \varepsilon), \varepsilon)| \le \max\{|R(t, y, \varepsilon)|, (t, y, \varepsilon) \in A \times K \times [-\varepsilon_0, \varepsilon_0]\} = N$. Then

$$\left| \int_{\bar{x}_\varepsilon}^t R\left(s, \varphi_X^2\left(s, \bar{x}_\varepsilon, \bar{x}_\varepsilon, y_0, \varepsilon\right), \varepsilon\right) ds \right| \le \int_{\bar{x}_\varepsilon}^t \left| R\left(s, \varphi_X^2\left(s, \bar{x}_\varepsilon, \bar{x}_\varepsilon, y_0, \varepsilon\right), \varepsilon\right) \right| ds$$

$$\le \int_A N ds = \mu(A)N,$$

which implies that

$$\int_{\bar{x}_\varepsilon}^t R\left(s, \varphi_X^2\left(s, \bar{x}_\varepsilon, \bar{x}_\varepsilon, y_0, \varepsilon\right), \varepsilon\right) ds = \mathcal{O}(1).$$

So

$$(10) \quad f(t) = \int_{\bar{x}_\varepsilon}^{t} F_1(s, \varphi_X^2(s, \bar{x}_\varepsilon, \bar{x}_\varepsilon, y_0, \varepsilon)) + F_2(s, \varphi_X^2(s, \bar{x}_\varepsilon, \bar{x}_\varepsilon, y_0, \varepsilon))ds + \mathcal{O}(\varepsilon).$$

From expression (10) and (4) we have that $f(z) - f_0(z) = \mathcal{O}(\varepsilon)$ for $z \in (\check{x}+T, \bar{x}+T)$. Indeed

$$\begin{aligned} |f(z) - f_0(z)| \leq \quad & \int_{\bar{x}}^{z} |F_1(s, \varphi_X^2(s, \bar{x}_\varepsilon, \bar{x}_\varepsilon, y_0, \varepsilon)) - F_1(s, y_0)|ds \\ & + \int_{\bar{x}}^{z} |F_2(s, \varphi_X^2(s, \bar{x}_\varepsilon, \bar{x}_\varepsilon, y_0, \varepsilon)) - F_2(s, y_0)|ds \\ & \pm (\bar{x}_\varepsilon - \bar{x}) + \mathcal{O}(\varepsilon), \end{aligned}$$

and

$$|F_i(s, \varphi_X^2(s, \bar{x}_\varepsilon, \bar{x}_\varepsilon, y_0, \varepsilon)) - F_i(s, y_0)| \leq L|\varphi_X^2(s, \bar{x}_\varepsilon, \bar{x}_\varepsilon, y_0, \varepsilon) - y_0| = \mathcal{O}(\varepsilon),$$

because F_i is L-Lipschitz continuous.

Thus we have that $f(z) = f_0(z) + \mathcal{O}(\varepsilon)$. From hypotheses and Lemma 4 we have assured, for $\varepsilon > 0$ sufficiently small, the existence of x_ε of Lemma 6, hence a sliding periodic orbit exists. $\qquad\square$

5. Proofs of Propositions E1 and E2

In this section we shall prove Proposition E1 by applying Theorem A. The proof of Proposition E2 is completely analogous.

Proof of Proposition E1. System (5) can also be written as

$$(11) \quad \begin{aligned} \dot{u}(t) &= -v + \frac{u}{2} - \text{sign}(h(u,v))\frac{u}{2} - \varepsilon\left(\frac{a\,u}{2} + \text{sign}(h(u,v))\frac{a\,u}{2}\right), \\ \dot{v}(t) &= u + \frac{v}{2} - \text{sign}(h(u,v))\frac{v}{2} + \varepsilon\left(\frac{b}{2} + \text{sign}(h(u,v))\frac{b}{2}\right). \end{aligned}$$

Applying the change of variables $(u,v) = (-y\sin(x), y\cos(x))$, system (11) becomes

$$(12) \quad \begin{aligned} \dot{x}(t) &= \ 1 - \varepsilon\,(1 + \text{sign}(y-1))\,\frac{(b + a\,y\cos(x))\sin(x)}{2y}, \\ \dot{y}(t) &= \ (1 - \text{sign}(y-1))\,\frac{y}{2} + \varepsilon\,(1 + \text{sign}(y-1))\,\frac{b\cos(x) - a\,y\sin^2(x)}{2}. \end{aligned}$$

Taking x as the new independent variable, the system (12) can be reduced to the form of (1), namely

$$(13) \quad \begin{aligned} \frac{dy}{dx}(x) = \ & (1 - \text{sign}(y-1))\,F_3(x,y) \\ & + \varepsilon\,(F_1(x,y) + \text{sign}(y-1)F_2(x,y)) + \mathcal{O}(\varepsilon^2), \end{aligned}$$

where

$$F_3(x, y) = \frac{y}{2},$$

$$F_1(x, y) = F_2(x, y) = \frac{b\cos(x) - a\,y\sin^2(x)}{2}.$$

Now, taking

$$\check{x} = \arctan\left(\sqrt{\frac{2b}{\sqrt{4a^2 + b^2} - b}}\right) \quad \text{and} \quad \bar{x} = 2\pi - \arctan\left(\sqrt{\frac{2b}{\sqrt{4a^2 + b^2} - b}}\right),$$

and assuming $a > 0$ and $b > 0$ we have that $F_1(\bar{x}) + F_2(\bar{x}) = 0$ and $F_1(\check{x}) + F_2(\check{x}) = 0$. It is easy to see that $F_1(x, 1) + F_2(x, 1) < 0$ for every $x \in (\check{x}, \bar{x})$, $F_1(x, 1) + F_2(x, 1) > 0$ for every $x \in (\bar{x}, \check{x} + 2\pi)$, $F_1(x, 1) - F_2(x, 1) = a > 0$ for every $x \in \mathbb{R}$, since $a > 0$, $b > 0$ and $c > 0$. Moreover

$$\partial_x F_1(\check{x}, 1) + \partial_x F_2(\check{x}, 1) = -(\partial_x F_1(\bar{x}, 1) + \partial_x F_2(\bar{x}, 1)) =$$

$$\frac{-\sqrt{2}c}{c + \sqrt{4b^2 + c^2}} \left(\sqrt{\sqrt{4b^2 + c^2} + c^2} + \frac{2b}{c}\sqrt{\sqrt{4b^2 + c^2} - c^2}\right).$$

Hence $\partial_x F_1(\bar{x}, 1) + \partial_x F_2(\bar{x}, 1) < 0$ and $\partial_x F_1(\check{x}, 1) + \partial_x F_2(\check{x}, 1) > 0$.

Computing the averaging function (4) with respect to (13) we have
(14)

$$\begin{aligned}
f_0(z) &= \int_{\bar{x}}^{z} b\cos(x) - a\sin^2(x)\,dx \\
&= \frac{2b\sin(z) + a\cos(z)\sin(z) - a\,z}{2} - \frac{2b\sin(\bar{x}) + a\cos(\bar{x})\sin(\bar{x}) - a\,\bar{x}}{2}.
\end{aligned}$$

Therefore by applying Theorem A, for a zero $z_0 \in (\check{x} + 2\pi, \bar{x} + 2\pi)$ of $f_0(z)$ such that $f'(z_0) \neq 0$, there exists a sliding periodic solution $Y(x, \varepsilon)$ of the differential equation (11) such that $Y(\bar{x}, \varepsilon) \to 1$ when $\varepsilon \to 0$, which implies in existence of a periodic solution $(x(t, \varepsilon), y(t, \varepsilon)) = (x(t, \varepsilon), Y(x(t, \varepsilon), \varepsilon))$ of the system (12) such that $(x(0, \varepsilon), y(0, \varepsilon)) \to (\bar{x}, 1)$ when $\varepsilon \to 0$. Hence,

$$(u(t, \varepsilon), v(t, \varepsilon)) = (-y(t, \varepsilon)\cos(x(t, \varepsilon)), y(t, \varepsilon)\sin(x(t, \varepsilon)))$$

is a sliding periodic solution of the system (5) such that $(u(0, \varepsilon), v(0, \varepsilon)) \to (-\sin(\bar{x}), \cos(\bar{x}))$ when $\varepsilon \to 0$.

Observe that for the averaged function (14), $f_0(\check{x} + 2\pi) > 0$ and $f_0(\check{x} + 2\pi) = -a\pi$. Since $a > 0$, there exists a zero $z_0 \in (\check{x} + 2\pi, \bar{x} + 2\pi)$ of the equation $f_0(z) = 0$. Moreover

$$f'(z) = b\cos(z) - a\sin^2(z) > 0$$

for $a, b > 0$ and $z \in (\check{x} + 2\pi, \bar{x} + 2\pi)$. Thus $f(z_0) = 0$ and $f'(z_0) \neq 0$, which concludes the proof of proposition. $\qquad\square$

APPENDIX: DEFINITION OF SOLUTIONS AND STRUCTURAL STABILITY AT THE DISCONTINUITY

Following [1], we summarize here a few of the basic concepts of discontinuous vector fields that are used in this paper.

We assume that discontinuity only appears in a differential submanifold Σ, which can be given as $\Sigma = h^{-1}(0) \cap U$ for a certain C^1 function h which has 0 as a regular value. Then the curve Σ splits U in two open sets

$$\Sigma^+ = \{(x, y) \in U : h(x, y) > 0\} \quad \text{and} \quad \Sigma^- \{(x, y) \in U : h(x, y) < 0\}.$$

We consider the following piecewise continuous vector field

(15)
$$Z(x, y) = \begin{cases} X(x, y), & \text{if } (x, y) \in \Sigma^+, \\ Y(x, y), & \text{if } (x, y) \in \Sigma^-, \end{cases}$$

which we denote concisely as $Z = (X, Y)_\Sigma$.

The curve Σ can be decomposed as the union of the closure of the regions:

$$\Sigma^c = \{x \in \Sigma : (Xh)(Yh)(x, y) > 0\};$$
$$\Sigma^e = \{x \in \Sigma : (Xh)(x, y) > 0 \ \& \ (Yh)(x, y) < 0\};$$
$$\Sigma^s = \{x \in \Sigma : (Xh)(x, y) < 0 \ \& \ (Yh)(x, y) > 0\}.$$

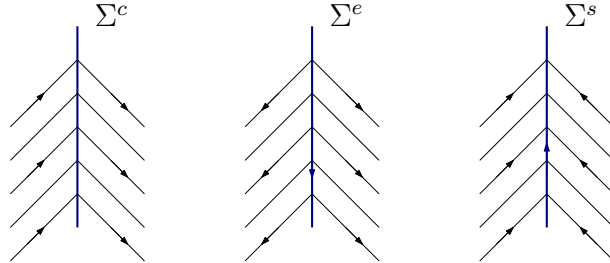Here $(Xh)(x, y) = \langle \nabla h(x, y), X(x, y) \rangle$ where $\langle \cdot, \cdot \rangle$ denotes the usual dot product of $\mathbb{R}^2$.

FIGURE 9.   Crossing region (Σ^c), escaping region (Σ^e) and sliding region (Σ^s).

For $p \in \Sigma^e \cup \Sigma^s$ we define the *sliding vector field* as

(16)
$$Z_s(p) = \frac{(Yh)(p)X(p) - (Xh)(p)Y(p)}{(Yh)(p) - (Xh)(p)}.$$

Let $\varphi_W(t, p) : I_p \times U \to \mathbb{R}^2$ be the flow induced by a vector field W such that $\varphi(0, p) = p$. Here, I_p is the maximal interval for which φ_W is defined. The local

trajectory of the system (15) of an orbit passing through $p \in U$ is given by the following definition.

Definition 1. *The local trajectory (or orbital solution) of a Filippov vector field of the form* (15) *through a point p is defined as follows:*

- *for $p \in \Sigma^+$ or $p \in \Sigma^-$ such that $X(p) \neq 0$ or $Y(p) \neq 0$ respectively, the trajectory is given by $\varphi_Z(t,p) = \varphi_X(t,p)$ or $\varphi_Z(t,p) = \varphi_Y(t,p)$, for $t \in I_p \subset \mathbb{R}$;*
- *for $p \in \Sigma^c$ such that $(Xh)(p), (Yh)(p) > 0$ and taking the origin of time at p, the trajectory is defined as $\varphi_Z(t,p) = \varphi_Y(t,p)$ for $t \in I_p \cap \{t < 0\}$ and $\varphi_Z(t,p) = \varphi_X(t,p)$ for $t \in I_p \cap \{t > 0\}$. For the case $(Xh)(p), (Yh)(p) < 0$ the definition is given by reversing time;*
- *for $p \in \Sigma^e \cup \Sigma^s$ such that $Z_s(p) \neq 0$, $\varphi_Z(t,p) = \varphi_{Z_s}(t,p)$ for $t \in I_p \subset \mathbb{R}$;*
- *for $p \in \partial\Sigma^c \cup \partial\Sigma^s \cup \partial\Sigma^e$ such that the definitions of trajectories for points in Σ in both sides of p can be extended to p and coincide, the orbit through p is the limiting orbit.*
- *any other points $\varphi(t,p) = p$ for all $t \in \mathbb{R}$ are not regular, for example irregular tangency points.*

Definition 2. *The following are generic singularities of the Filippov vector field* (15):

- *$p \in \Sigma^\pm$ such that $X(p) = 0$ and $Y(p) = 0$ respectively;*
- *$p \in \Sigma^s \cup \Sigma^e$ such that $Z_s(p) = 0$, called a pseudoequilibrium;*
- *$p \in \partial\Sigma^c \cup \partial\Sigma^s \cup \partial\Sigma^e$, that is (regular and singular) tangency points.*

For planar Filippov vector fields, there exist the following generic singularities which are all distinguished singularities (see definition 2).

(1) A regular-fold point is some $p \in \Sigma$ such that $Xf(p) = 0$ and $X^2f(p) \neq 0$ and $Yf(p) \neq 0$ or points such that $Xf(p) = 0$ and $Y^2f(p) \neq 0$ and $Xf(p) \neq 0$.
(2) A hyperbolic fixed point of the sliding vector field is some $p \in \Sigma^s \cup \Sigma^e$ such that $X(p)||Y(p)$ and hence $Z_s(p) = 0$. Moreover, we impose the generic condition $Z_s(p) \neq 0$.

Definition 3. *Two Filippov vector fields $Z = (X,Y)_\Sigma : U \to \mathbb{R}^2$ and $\check{Z} = (\check{X}, \check{Y})_{\check{\Sigma}} : \check{U} \to \mathbb{R}^2$ are Σ-equivalent if there exists an orientation preserving homeomorphism $H : U \to \check{U}$ which sends $\Sigma \subset U$ to $\check{\Sigma} \subset \check{U}$ and sends orbits of Z to orbits of $\check{Z}$.*

The definition of Σ-equivalence gives rise to the concepts of Σ-structural stability, see [1].

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